J.T. Lyczak, January 2016

Spectral sequences

A useful tool in homological algebra is the theory of spectral sequences. The purpose of this text is to introduce the reader to the subject and proofs are generally omitted for clearity. At the end the reader will be hopefully be able to work with spectral sequences as they arise in practice.
For a thorough treatise on spectral sequences one can consult at any of the many books on homological algebra. Below the book by [Weibel] is used as a reference for theorems without proofs.

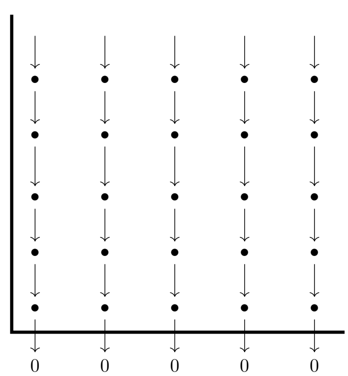
1 Homological spectral sequences

Definition 1. A *homological spectral sequence* (or simply *spectral sequences*) in an abelian category (for example modules over a fixed commutative ring) consists of

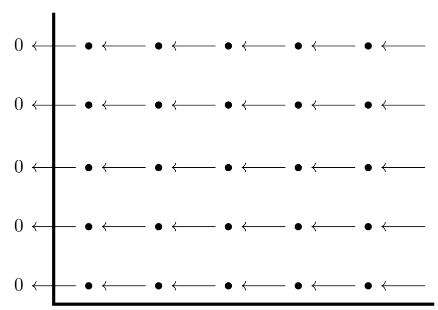
- a non-negative integer a;
- objects E^r_{pq} for every $r \geq a$ indexed by integers p and q. We assume that $E^r_{pq} = 0$ for negative p or q;
- morphisms $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$ for all $r \geq a$ and all p and q, such that $d^r \circ d^r = 0$;
- isomorphisms between E^{r+1}_{pq} and the homology at E^r_{pq} under the complex given by the morphisms d^r.

One can think of E^r as a family of complexes whose homology groups are the objects of E^{r+1}:

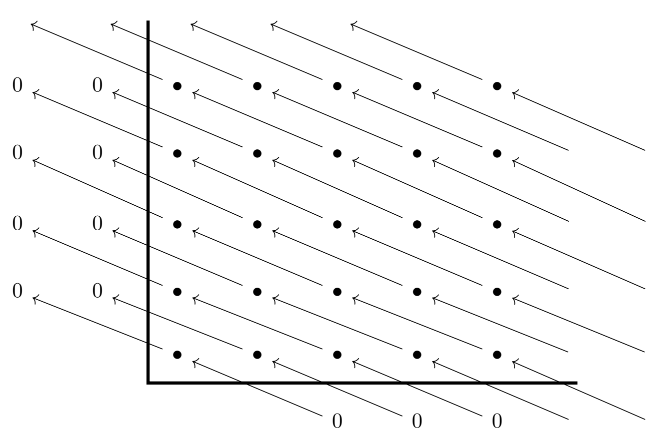
E^0 :



E^1 :



E^2 :



Note that the objects E^{r+1} can be recovered from the rth page, but the differentials d^{r+1} cannot and are part of the given structure.

Let us now fix p and q for a moment and look at the objects at position (p,q) for increasing r. First note that each E^{r+1}_{pq} is a subquotient of E^r_{pq}. One also sees that if $r > \max(p, q+1)$, then E^r_{pq} lives in the complex $0 \xrightarrow{d^r} E^r_{pq} \xrightarrow{d^r} 0$.

So we find $E^r_{pq} \cong E^{r+1}_{pq} \cong E^{r+2}_{pq} \cong \ldots$. Hence, for large enough r the objects on position (p,q) stabilize. We will write E^∞_{pq} for this object.

The usefulness of spectral sequences lies in convergence, which we will now define.

Definition 2. Let H_n be an object for every integer $n \geq 0$. We say that a homological spectral sequence E^a_{pq} converges to H_{p+q}, notation

$$E^a_{pq} \Rightarrow H_{p+q},$$

if each H_n has a filtration

$$0 = (H_n)_{-1} \subseteq (H_n)_0 \subseteq (H_n)_1 \subseteq \ldots \subseteq (H_n)_{n-1} \subseteq (H_n)_n = H_n,$$

and we have isomorphisms

$$E^\infty_{pq} \cong (H_{p+q})_p/(H_{p+q})_{p-1}.$$

This means that the filtration on H_n allows one to determine the objects of E^∞ with $p+q=n$.

$$E^\infty: \quad \begin{array}{c|cccccc}
n & (H_n)_0/(H_n)_{-1} \cong (H_n)_0 & & & & & \\
n-1 & & (H_n)_1/(H_n)_0 & & & & \\
n-2 & & & (H_n)_2/(H_n)_1 & & & \\
\vdots & & & & \ddots & & \\
1 & & & & & (H_n)_{n-1}/(H_n)_{n-2} & \\
q=0 & & & & & & (H_n)_n/(H_n)_{n-1} \cong H_n/(H_n)_{n-1} \\
\hline
 & p=0 & 1 & 2 & \ldots & n-1 & n
\end{array}$$

Convergence of spectral sequences is particular useful when there are a lot of trivial objects on E^∞. Let us first consider the following extreme case.

Theorem 1. *Fix an integer n and suppose that we have a converging homological spectral sequence*

$$E^r_{pq} \Rightarrow H_{p+q}.$$

Then $E^\infty_{pq} = 0$ on the diagonal $p+q=n$ if and only if $H_n = 0$.

By far the most applications of spectral sequences occur in a slightly less specific situation named in the following definition.

Definition 3. A spectral sequence is said to *collapse* at E^s $(s \geq a)$ if E^s consists of only trivial objects outside of a single row or column. In this case we have $E^s \cong E^\infty$ (at least for $s \geq 2$).

For a converging spectral sequence which collapses we have the following relation between the ∞-page and the limit.

Theorem 2. *Suppose that a converging spectral sequence*

$$E^r_{pq} \Rightarrow H_{p+q}$$

collapses at E^s ($s \geq 2$). Then H_n is isomorphic to the unique non-zero E^s_{pq} with $p+q=n$.

In other situations there is still something to say, but we do not get isomorphisms between H_n and objects of the spectral sequence any more. We will just be able to say something on H_n up to extension, in the sense made clear by the following exercises.

Exercise 1 ([Weibel], Ex 5.2.1 (Corrected))**.** Suppose that a spectral sequence sequence converging to H_* has $E^2_{pq} = 0$ unless $p = 0, 1$. Show that there are exact sequences

$$0 \longrightarrow E^2_{0n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

Generalize to two non-zero columns at $p = k, l \geq 0$.

Exercise 2 ([Weibel], Ex 5.2.2)**.** Suppose that a spectral sequence sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. Show that there is a long exact sequence

$$\ldots \longrightarrow H_{p+1} \longrightarrow E^2_{p+1,0} \xrightarrow{d^2} E^2_{p-1,1} \longrightarrow H_p \longrightarrow E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \longrightarrow H_{p-1} \longrightarrow \ldots.$$

Generalize to two non-zero rows at $q = k, l \geq 0$.

Exercise 3 (Five term sequence)**.** Consider a converging spectral sequence

$$E^2_{pq} \Rightarrow H_{p+q}.$$

Then we have an exact sequence of a the first terms of low degree

$$H_2 \longrightarrow E^2_{20} \xrightarrow{d^2} E^2_{01} \longrightarrow H_1 \longrightarrow E^2_{10} \longrightarrow 0.$$

Exercise 4 (Edge morphisms)**.** Show that for a spectral sequences and all r larger than 2 (and a), we have natural maps

$$E^r_{0n} \twoheadrightarrow E^\infty_{0n} \text{ and } E^\infty_{n0} \subseteq E^r_{n0}.$$

If the spectral sequence converges to H_n we get maps

$$E^r_{0n} \twoheadrightarrow E^\infty_{0n} \subseteq H_n \text{ and } H_n \twoheadrightarrow E^\infty_{n0} \subseteq E^r_{n0}.$$

These maps are called the *edge morphisms.*

2 Construction of homological spectral sequences

We have seen some terminology and theory concering spectral sequences. Now we will give a few ways to produce spectral sequences. The first one being the most general.

2.1 Spectral sequence of a filtered complex

Let $C_\bullet$ be a complex concentrated in non-negative degrees and let $(C)_n$ be a filtration of the complex which is canonically bounded, i.e. $0 = (C_n)_{-1} \subseteq (C_n)_0 \subseteq \ldots \subseteq (C_n)_{n-1} = (C_n)_n = C_n$ for all n. We can depict this as

$$
\begin{tikzcd}
C : & \cdots \arrow[r] & C_3 \arrow[r, "d_3"] \arrow[d, equal] & C_2 \arrow[r, "d_2"] \arrow[d, equal] & C_1 \arrow[r, "d_1"] \arrow[d, equal] & C_0 \arrow[r] \arrow[d, equal] & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
(C)_2 : & \cdots \arrow[r] & (C_3)_2 \arrow[r] \arrow[u, hook] & (C_2)_2 \arrow[r] \arrow[u, equal] & C_1 \arrow[r] \arrow[u, equal] & C_0 \arrow[r] \arrow[u, equal] & 0 \\
(C)_1 : & \cdots \arrow[r] & (C_3)_1 \arrow[r] \arrow[u, hook] & (C_2)_1 \arrow[r] \arrow[u, hook] & (C_1)_1 \arrow[r] \arrow[u, equal] & C_0 \arrow[r] \arrow[u, equal] & 0 \\
(C)_0 : & \cdots \arrow[r] & (C_3)_0 \arrow[r] \arrow[u, hook] & (C_2)_0 \arrow[r] \arrow[u, hook] & (C_1)_0 \arrow[r] \arrow[u, hook] & (C_0)_0 \arrow[r] \arrow[u, equal] & 0 \\
(C)_{-1} : & \cdots \arrow[r] & 0 \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}
$$

in which all the horizontal maps are restriction of the differenatials in the original complex. So all rows are complexes too.

In each column beneath C_n we have exactly $n+1$ interesting quotients $(C_n)_p/(C_n)_{p-1}$. Let us put these on the diagonal in a grid as we did for the filtration of the family $H_\bullet$ above.

$$
E^0 : \quad
\begin{array}{c|ccccc}
4 & (C_4)_0 & & & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & (C_3)_0 & (C_4)_1/(C_4)_0 & & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & (C_2)_0 & (C_3)_1/(C_3)_0 & (C_4)_2/(C_4)_1 & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & (C_1)_0 & (C_2)_1/(C_2)_0 & (C_3)_2/(C_3)_1 & (C_4)_3/(C_4)_2 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
q=0 & C_0 & C_1/(C_1)_0 & C_2/(C_2)_1 & C_3/(C_3)_2 & C_4/(C_4)_3 \\
\hline
 & p=0 & 1 & 2 & 3 & 4
\end{array}
$$

Here the horizontal maps for a fixed p come from the complex $(C)_p$. We can extend this zeroth page into a spectral sequence. The construction, like the construction of the connecting homomorphisms or the snake map, is by direct definition and not very enlightening. We will just state the existence of the differentials on subsequent pages and its convergence properties.

Theorem 3. *Let $(C)_k$ be a canonically bounded filtration on a complex $C_\bullet$. There exists a natural spectral sequence starting with $E^0_{pq} = (C_{p+q})_p/(C_{p+q})_{p-1}$ such that it converges to the homology of $C_\bullet$:*

$$E^0_{pq} \Rightarrow H_{p+q}(C).$$

Exercise 5. Verify the above theorem for the filtration $(C)_n = C_\bullet$ for all $n \geq 0$.
Do the same for the filtration

$$(C_k)_l = \begin{cases} 0 & \text{for } l < k; \\ C_k & \text{for } l \geq k. \end{cases}$$

Now change $(C_1)_0$ into C_1. What happens?

Exercise 6. Consider the complex

$$\cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} 0$$

of abelian groups. Define the filtration

$$\begin{tikzcd}
\cdots \arrow[r] & R_2 \arrow[r, "\cdot 2"] \arrow[d, equal] & R_1 \arrow[r, "\cdot 2"] \arrow[d, equal] & R_0 \arrow[r] \arrow[d, equal] & 0 \\
& \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \\
\cdots \arrow[r] & R_{22} \arrow[r, "\cdot 2"] & R_{12} \arrow[r, "\cdot 2"] & R_{02} \arrow[r] & 0 \\
\cdots \arrow[r] & 2R_2 \arrow[u, hook] \arrow[r, "\cdot 2"] & R_{11} \arrow[u, equal] \arrow[r, "\cdot 2"] & R_{01} \arrow[u, equal] \arrow[r] & 0 \\
\cdots \arrow[r] & 0 \arrow[u, hook] \arrow[r] & 2R_1 \arrow[u, hook] \arrow[r, "\cdot 2"] & R_{00} \arrow[u, equal] \arrow[r] & 0 \\
\cdots & 0 \arrow[u, hook] & 0 \arrow[u, hook] & 0 \arrow[u, hook] &
\end{tikzcd}$$

where each R_n and R_{pq} is a copy of $\mathbb{Z}/4\mathbb{Z}$.
The indices are simply to keep track of our objects. Every object E^r_{pq} is a subquotient of objects in the filtration. So make sure every object you compute below is written as either R_{pq}, $2R_p$, $R_{pq}/2R_p$ or 0.

a) Determine E^0 and note that the differentials come directly from the diagram above. Compute E^1 while keeping track of your indices.

b) Keeping your indices in mind, remark that there is only one obvious choice for the morphisms d^1. Use this to compute E^2.

c) Determine the morphisms d^2 (again, there is only one natural choice). Now compute E^3 and show that this is actually E^∞. Conclude that you just defined a spectral sequence which converges to the homology of the complex above.

If you did the above exercise, then you probably tried to compute the differential $d^2 : E^2_{20} \to E^2_{01}$. For this you just had to notice that the image of the morphism $R_{22} \to R_{12}$ even lies in $2R_1 \hookrightarrow R_{12}$. This is exactly what happens in the construction of this spectral sequence for general filtered complexes, which uses the intermediate objects

$$A^r_{p,n-p} := \{x \in (C_n)_p \mid d_n(x) \text{ lies in } (C_{n-1})_{p-r} \subseteq (C_{n-1})_p\}\,.$$

Since $(C_n)_p$ is part of C_n and $(C_{n-1})_{p-r}$ is smaller than $(C_{n-1})_p$, we can think of $(C_n)_p$ becoming closer to C_n for large p and $(C_{n-1})_{p-r}$ going to 0. So we sometimes say the elements of $A^r_{p,n-p}$ approximate the cycles in C_n and one can define E^r_{pq} as a quotient of these A^r_{pq}. For example the definition

$$E^r_{pq} := \frac{A^r_{pq}}{d_{p+q+1}(A^{r-1}_{p+r-1,q-r+2}) + A^{r-1}_{p-1,q+1}}$$

works for all $r \geq 1$. The morphisms d^r are now simply restrictions of the maps d_n in the orginal complex.

$$\begin{tikzcd}[column sep=small]
(C_{n+1})_{p+(r-1)} & \supseteq & A^{r-1}_{p+r-1,q-r+2} \arrow[r] \arrow[ddrrr, dashed, "d_{n+1}"] & (C_n)_{p+(r-1)} & & & & & \\
\vdots \arrow[u, hook] & & & \vdots \arrow[u, hook] & & & & & \\
(C_{n+1})_p \arrow[rrr] \arrow[u, hook] & & & (C_n)_p \arrow[u, hook] & \supseteq & A^r_{p,q} \arrow[r] \arrow[dddrrr, dashed, "d_n"] & (C_{n-1})_p & & \\
& & & (C_n)_{p-1} \arrow[u, hook] & \supseteq & A^{r-1}_{p-1,q+1} \arrow[u, hook] \arrow[ddrrr, dashed, "d_n"] & \vdots \arrow[u, hook] & & \\
& & & & & & \vdots \arrow[u, hook] & & \\
& & & & & & (C_{n-1})_{p-r} \arrow[u, hook] & \supseteq & A^r_{p-r,q+r-1}
\end{tikzcd}$$

Exercise 7. Use the above diagram to conclude that we have maps $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$ and that these maps form complexes.

The proof that the homology on the rth page yields the objects of the $(r+1)$th page is rather tedious, but straight forward. An efficient proof can be found in section 5.4 in [Weibel]. In section 5.5 of this book one can also find a proof that such a spectral sequence associated to a complex with a grading, converges to the homology of the complex.

2.2 Spectral sequences of a double complex

Many interesting spectral sequences are special examples of the sequences defined in the previous chapter. For example, using this we can define two spectral sequence associated to a double complex. First we will need the following definition.

Definition 4 (Total complex)**.** Let C_{pq} be a first quadrant total complex, i.e. $C_{pq} = 0$ if $p < 0$ or $q < 0$. We define the *total complex* by

$$(\operatorname{Tot} C)_n = \bigoplus_{p+q=n} C_{pq}.$$

The morphism $d_n : (\operatorname{Tot} C)_n \to (\operatorname{Tot} C)_{n-1}$ maps an element $(c_{pq})_{p+q=n} \in (\operatorname{Tot} C)_n$ to the element $(d_h(c_{p,q}) + d_v(c_{p+1,q-1}))_{p+q=n} \in (\operatorname{Tot} C)_{n-1}$.

Basically, each element in $C_{p,q} \subseteq (\operatorname{Tot} C)_{p+q}$ gets send using both the horizonal and the veritcal maps of $C_{\bullet,\bullet}$ to both possible summands in $(\operatorname{Tot} C)_{p+q-1}$.

We have a natural filtration on a double complex given by the horizontal truncations:

$$\mathrm{tr}^{\mathrm{hor}}(C)_{\leq 0} = \begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
C_{03} \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{02} \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{01} \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{00} & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & {} \arrow[l]
\end{tikzcd} \qquad \mathrm{tr}^{\mathrm{hor}}(C)_{\leq 1} = \begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
C_{03} \arrow[d] & C_{13} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{02} \arrow[d] & C_{12} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{01} \arrow[d] & C_{11} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & {} \arrow[l] \\
C_{00} & C_{10} \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & {} \arrow[l]
\end{tikzcd}$$

$$
\operatorname{tr}^{\text{hor}}(C)_{\le 2} = \qquad \begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{03} & \longleftarrow & C_{13} & \longleftarrow & C_{23} & \longleftarrow & 0 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{02} & \longleftarrow & C_{12} & \longleftarrow & C_{22} & \longleftarrow & 0 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{01} & \longleftarrow & C_{11} & \longleftarrow & C_{21} & \longleftarrow & 0 & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{00} & \longleftarrow & C_{10} & \longleftarrow & C_{20} & \longleftarrow & 0 & \longleftarrow &
\end{array}
$$

The total complexes of these truncations of $C_{\bullet,\bullet}$ give a canonically bounded filtration on the total complex of $C_{\bullet,\bullet}$:

$$C_{0n} \subseteq C_{0n} \oplus C_{1,n-1} \subseteq \ldots \subseteq C_{0n} \oplus C_{1,n-1} \oplus \ldots \oplus C_{n-1,1} \oplus C_{n,0} = (\operatorname{Tot} C)_n.$$

We have the following result about the spectral sequence coming from this filtration on the total complex:

Theorem 4. *There exists a spectral sequence E^a_{pq} for $a \ge 0$ with*

- *the zeroth page equal to the original double complex: $E^0_{pq} = C_{pq}$;*
- *the morphisms on the zeroth page d^0 are the vertical maps d_v in $C_{\bullet,\bullet}$;*
- *the morphisms d^1 on the first page are naturally induced by the horizontal morphisms d_h of the original double complex.*

This explains the suggestive notation

$$E^2_{pq} = H^{hor}_p(H^{ver}_q(C))$$

and we have the convergence

$$H^{hor}_p(H^{ver}_q(C)) \Rightarrow H_{p+q}(\operatorname{Tot} C).$$

We could easily have used the vertical truncations of the double complex. This gives a differrent spectral sequence with the same limit.

Theorem 5. *For every double complex $C_{\bullet,\bullet}$ we have a spectral sequence satisfying*

$$E^2_{pq} = H^{ver}_p(H^{hor}_q(C)) \Rightarrow H_{p+q}(\operatorname{Tot} C).$$

Note that the we still first compute the homology at the qth position and then at the pth position. This makes E^2_{pq} actually into a subquotient of C^2_{qp}.

Using both these spectral sequences at the same time can give a lot of information.

Example 1 (Snake lemma)**.** Consider the map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}
$$

There is a natural exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0.$$

We will prove this important lemma using the machinery of spectral sequences.

Consider the first quadrant double complex, whose lower left corner is given by

$$\begin{tikzcd}
C \arrow[d, "h"] & B \arrow[l] \arrow[d, "g"] & A \arrow[l] \arrow[d, "f"] & 0 \arrow[l] \\
C' & B' \arrow[l] & C' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

All objects and maps which are not shown are assumed to be zero. If we apply theorem 5 to this double complex, we see that above we have the following zeroth and first page of the spectral sequence converging to $H_n(\operatorname{Tot} C)$:

$$E^0 : \begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A' \arrow[d] & A \arrow[d] \\
B' \arrow[d] & B \arrow[d] \\
C' & C
\end{tikzcd} \quad \text{and} \quad E^1 : \begin{tikzcd}
0 & 0 \arrow[l] \\
0 & 0 \arrow[l] \\
0 & 0 \arrow[l] \\
0 & 0 \arrow[l]
\end{tikzcd}$$

So we conclude that in fact $H_\bullet(\operatorname{Tot} C) = 0$.
Now let use compute the first pages of the spectral sequence described in theorem 4. The zeroth page is simply the original double complex (just forget the horizontal maps). For the first page we find

$$E^1 : \begin{tikzcd}
\ker h & \ker g \arrow[l] & \ker f \arrow[l] & 0 \arrow[l] \\
\operatorname{coker} h & \operatorname{coker} g \arrow[l] & \operatorname{coker} f \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

These sequence are not necessarily exact, but we know that their homology groups are given by E^2. The second page looks like

$$E^2 : \begin{tikzcd}
 & & 0 & & & & \\
 & & * \arrow[ull] & \boxed{*} \arrow[ull] & \boxed{*} \arrow[ull] & 0 & \\
 & & \boxed{*} \arrow[ull] & \boxed{*} \arrow[ull] & * \arrow[ull] & 0 \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

However, we see that the boxed objects on E^2 do not on subsequent pages. So the boxed objects must be zero, since the limit is zero. So the two sequences on the first page are exact (except possibly at $\ker h$ and $\operatorname{coker} f$):

$$0 \longleftarrow \operatorname{coker} h \longleftarrow \operatorname{coker} g \longleftarrow \operatorname{coker} f \quad \text{and} \quad \ker h \longleftarrow \ker g \longleftarrow \ker f \longleftarrow 0.$$

The kernel of the far right map in the first sequence and the cokernel of the far left map in the second sequence are exactly the two remaining non-zero objects $E^2_{20} \to E^2_{01}$ on E^2. We will prove that the map between them is in fact an isomorphism. The kernel and cokernel are respectively E^3_{20} and E^3_{01} which are exactly E^∞_{20} and E^∞_{01}, which must be zero since the limit of the spectral sequence is zero.

So we can splice the two exact sequence together to get a longer exact sequence

$$0 \longleftarrow \operatorname{coker} h \longleftarrow \operatorname{coker} g \longleftarrow \operatorname{coker} f \longleftarrow \ker h \longleftarrow \ker g \longleftarrow \ker f \longleftarrow 0.$$

This shows that the construction of the higher morphisms in the spectral sequences is related to the connection homomorphisms, as the connecting homomorphism in the snake lemma is simply a morphism d^2 on E^2 of the spectral sequence described in theorem 4.

The same trick can be applied to the following classical results in homological algebra.

Exercise 8 (Acyclic assembly lemma)**.** Let $C_{\bullet,\bullet}$ be a double complex in the first quadrant. If all either all rows or all columns are exact, then so is the complex $\mathrm{Tot}(C)$.

Exercise 9 (The long exact sequence of homology)**.** Let

$$0 \longrightarrow C_\bullet \longrightarrow D_\bullet \longrightarrow E_\bullet \longrightarrow 0$$

be an exact sequence of complexes. Prove that we have the following exact sequence

$$\dots \longrightarrow H_n(C) \longrightarrow H_n(D) \longrightarrow H_n(E) \longrightarrow H_{n-1}(C) \longrightarrow H_{n-1}(D) \longrightarrow H_{n-1}(E) \longrightarrow \dots$$

Exercise 10 (The five lemma)**.** Consider the map of exact rows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

a) Prove that if f, g, i and j are isomorphisms, so is h.

b) Use your proof to show how to relax the conditions on f, g, i and j, without changing the conclusion.

Exercise 11 ($\mathrm{Tor}_\bullet(A,B) = \mathrm{Tor}_\bullet(B,A)$)**.** Let R be a commutative ring and consider elements A and B in the abelian category of R-modules, where R is a commutative ring. Choose projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$. Consider the double complex $(P \otimes Q)_{pq} = P_p \otimes Q_q$ to prove that

$$\mathrm{Tor}_\bullet(A,B) = \mathrm{Tor}_\bullet(B,A).$$

2.3 Grothendieck spectral sequence

There is a rather general constructing of spectral sequences relating the higher derived functors of a composition of functors to the higher derived functors of the respective functors. We will omit the proof, which can be found in [Weibel].

Theorem 6 (Grothendieck spectral sequence)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two right exact additive functors, such that F maps projective objects in $\mathcal{A}$ to a G-acyclic object in $\mathcal{B}$. Then for any object $A \in \mathcal{A}$ there exists a spectral sequence*

$$E^2_{pq} = L^pG \circ L^qF(A)$$

converging to

$$L^{p+q}(G \circ F)(A).$$

Note that it makes sense to talk about the higher derived functors of $G \circ F$ as it is right exact, since F and G both are.

This spectral sequence is extremely useful if it collapses at one of the early pages, but we always have the exact sequence of remt of low degree:

$$L^2(G \circ F)A \longrightarrow (L^2G)(FA) \longrightarrow G(L^1F(A)) \longrightarrow L^1(G \circ F)A \longrightarrow (L^1G)(FA) \longrightarrow 0.$$

Exercise 12. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive functors. Assume F is right exact and G is exact. Prove that for all $q \geq 0$ one has

$$L^q(G \circ F) = G \circ L^qF.$$

3 Cohomological spectral sequences

We will now look at the dual notion of homological spectral sequences. These are aptly named cohomological spectral sequences. Using all we did above we can define a cohomological spectral sequence in an abelian category $\mathcal{C}$ as an homological spectral sequence in the oppposite category $\mathcal{C}^{\mathrm{op}}$. If we consider the objects in $\mathcal{C}$ we get homological spectral sequences with all the arrows inverted. Let us, for completeness, state a definition like the one for homological spectral sequences.

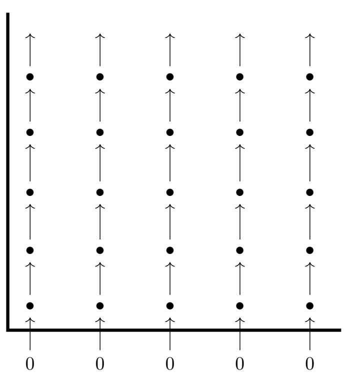
Definition 5. A *cohomological spectral sequence* in an abelian category (for example modules over a fixed ring) consists of

- a non-negative integer a;
- objects E_r^{pq} for every $r \geq a$ indexed by integers p and q. We assume that $E_r^{pq} = 0$ for negative p or q;
- morphisms $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$ for all $r \geq a$ and all p and q, such that $d_r \circ d_r = 0$;
- isomorphisms between E_{r+1}^{pq} and the cohomology at E_r^{pq} under the complex given by the morphisms d_r.

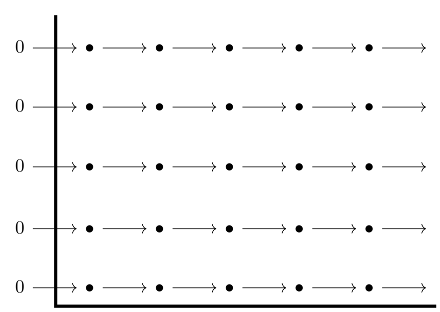
In case, no confusion can arise the term *spectral sequence* can refer to either a homological or a cohomological spectral sequence. In practice it will always be clear by the way the objects are indexed.

One can think of E_r as a family of cocomplexes whose cohomology groups are the objects of E_{r+1}:

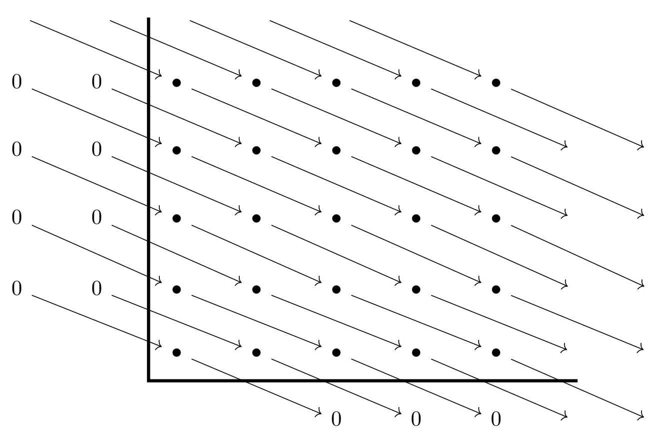
E_0 :



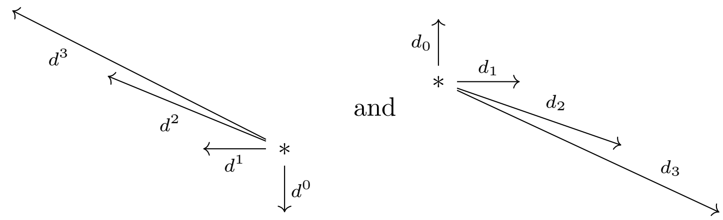
E_1 :



E_2 :



The difference in direction between the morphisms in a homological and cohomological spectral sequence are depicted in the following diagram:



Let us now turn to the notion of convergence for cohomological spectral sequences.

Definition 6. Let H^n be a family of objects indexed by integers $n \geq 0$. We say that a cohomological spectral sequence E_a^{pq} converges to H^{p+q}, notation

$$E_a^{pq} \Rightarrow H^{p+q},$$

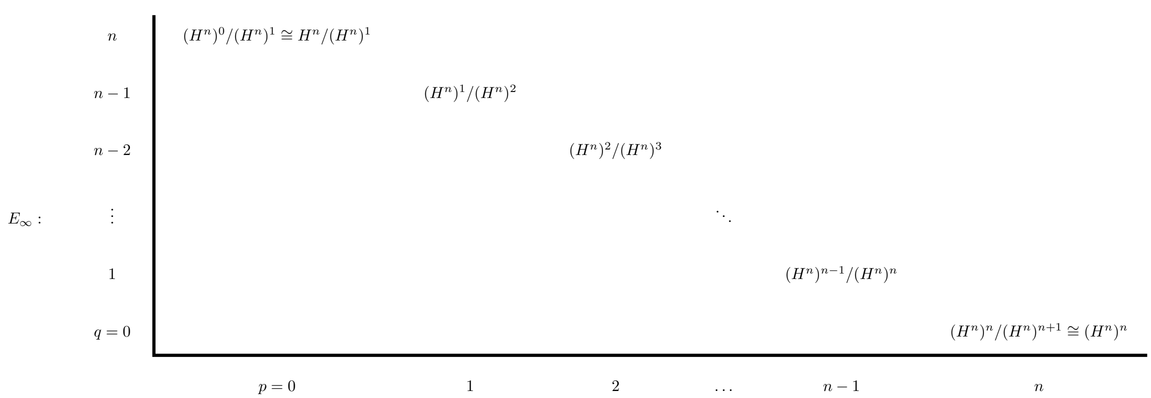
if each H^n has a filtration

$$0 = (H^n)^{n+1} \subseteq (H^n)^n \subseteq (H^n)^{n-1} \subseteq \ldots \subseteq (H^n)^1 \subseteq (H^n)^0 = H^n,$$

and we have isomorphisms

$$E_\infty^{pq} \cong (H^{p+q})^p / (H^{p+q})^{p+1}.$$

Again we can read off these quotient on the E_∞ page.



Exercise 13. Assume we have a converging cohomological spectral sequence $E_2^{pq} \Rightarrow H^{p+q}$. Make sense of the following concepts, we already saw for the homological spectral sequences.

a) $H^n = 0$ if and only $E_\infty^{pq} = 0$ for all $p + q = n$.

b) A single non-zero row or column at any page E_r allows one to read of the limit H^n as the unique non-zero object E_r^{pq} where $p + q = n$.

c) Two non-zero columns or two non-zero rows in E_r give a long exact sequence.

d) We have natural maps (also called edge maps):

$$E_r^{n0} \twoheadrightarrow E_\infty^{n0} \subseteq H^n \quad \text{and} \quad H^n \twoheadrightarrow E_\infty^{0n} \subseteq E_r^{0n}.$$

e) There exists an exact sequence of terms of low degree:

$$0 \longrightarrow E_2^{10} \longrightarrow H^1 \longrightarrow E_2^{01} \longrightarrow E_2^{20} \longrightarrow H^2.$$

4 Construction of cohomological spectral sequences

There are a few natural and important constructions of cohomological spectral sequences, similar to the case of homological spectral sequences. Some of the diagrams shown in the homological case are omitted here for brevity. The reader, however, is advised to produce them for the cohomological case as well.

4.1 Spectral sequences for filtrated cocomplexes and double cocomplexes

Consider a cocomplex $C^\bullet$ (concentrated in non-negative degree). A filtration $\ldots \subseteq (C)^2 \subseteq (C)^1 \subseteq (C)^0 = C^\bullet$ is said to be canonically bounded if the filtration of every object C^n in the complex is of the form $0 = (C^n)^{n+1} \subseteq (C^n)^n \subseteq \ldots \subseteq (C^n)^2 \subseteq (C^n)^1 \subseteq (C^n)^0 = C^n$.

Theorem 7. *Let $C^\bullet$ be a canonically bounded cocomplex. Then there exists a converging spectral sequence*

$$E_0^{pq} = (C^{p+q})^p/(C^{p+q})^{p+1} \Rightarrow H^{p+q}(C).$$

Note that the total complex of a double cocomplex $C^{\bullet,\bullet}$ (as always concentrated in non-negative degrees) is a cocomplex (again concentrated in non-negative degrees.) Again, this gives us two ways to produce two spectral sequences with the same limit using the previous theorem.

Theorem 8. *Let $C_{\bullet,\bullet}$ be a double cocomplex. Then there exist two cohomological spectral sequences*

$$E_2^{pq} = H^p_{hor}(H^q_{ver}(C)) \quad \text{and} \quad E_2^{pq} = H^p_{ver}(H^q_{hor}(C))$$

both converging to $H^{p+q}(\operatorname{Tot} C)$.

Again, note that for the first spectral sequence E_2^{pq} is a quotient of a subobject of C^{pq}, while in the second it is a subquotient of C^{qp}.

Exercise 14. Redo either the example or one of the exercises in the previous section using cohomological spectral sequences and the result of the previous exercise.

Exercise 15. Let $C_{\bullet,\bullet}$ be a double complex which is zero outside $0 \le p \le k$ and $0 \le q \le l$. Then $D^{pq} = C_{k-p,l-q}$ is a cocomplex which is zero outside of the same bounds. Prove that the two homological spectral sequences associated to the double complex and the two cohomological spectral sequences related to the cocomplex have the same limit and the same second page under a reindexing of the objects.
In fact, the entire homological and cohomological sequence are related to each other in this way.

4.2 Grothendieck spectral sequence for right derived functors

Let us look at the dual statement to theorem 6.

Theorem 9. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories, such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two left exact additive functors, such that F maps injective objects in $\mathcal{A}$ to a G-acyclic object in $\mathcal{B}$. Then for any object $A \in \mathcal{A}$ there exists a cohomological spectral sequence starting with*

$$E_2^{pq} = R^pG \circ R^qF(A)$$

converging to

$$R^{p+q}(G \circ F)(A).$$

We have the following exact sequence of low degree terms

$$0 \longrightarrow (R^1G)(FA) \longrightarrow R^1(G \circ F)A \longrightarrow G(R^1F(A)) \longrightarrow (R^2F)(GA) \longrightarrow R^2(G \circ F)A.$$

5 Examples of spectral sequences

We have seen some natural ways to construct spectral sequences. Each being a special case of the previous construction. We can now use the Grothendieck spectral sequences to construct some spectral sequences which are useful in algebra, topology and geometry.

Theorem 10 (Base change for Tor)**.** *Let $f : S \to R$ be a map of commutative rings. Then for every R-module A and S-module B we have a spectral sequence*

$$\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, S), B) \Rightarrow \mathrm{Tor}_{p+q}^R(A, B).$$

Similarly, we have a spectral sequence for the Ext functors.

Theorem 11 (Base change for Ext)**.** *Let $f : S \to R$ be a map of commutative rings. Then for every S-module A and R-module B we have a spectral sequence*

$$\mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(A, B)) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B).$$

For general sheafs one has interesting and useful spectral sequence related to sheafs.

Theorem 12 (Leray spectral sequences)**.** *Let $f : X \to Y$ be a continuous map of topological spaces. Then for any sheaf $\mathcal{F}$ of abelian groups on X we have a spectral sequence*

$$H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Confusingly, there is another spectral sequence also named after Leray.

Theorem 13 (Leray spectral sequences)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous maps of topological spaces. Then for any sheaf $\mathcal{F}$ of abelian groups on X we have a spectral sequence*

$$R^p g_*(R^q f_* \mathcal{F}) \Rightarrow R^{p+q}(g \circ f)_* \mathcal{F}.$$

In geometry, one can often relate the global functors to the local functors using spectral sequences. We have for example for the Ext functors the following result.

Theorem 14 (Local to global Ext spectral sequences)**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of modules over a ringed space $(X, \mathcal{O}_X)$. Then we have a spectral sequence*

$$H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Exercise 16. In each of the previous examples, find the two functors which make them into Grothendieck spectral sequences. Also, determine for each example whether is concerns a homological or a cohomological spectral sequence.