Hyperbolic Geometry of knots – Erik Visse

February 1st, 2016

These are the knotes from the seminar on knot theory in Leiden in the spring of 2016. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~lyczakjt/seminar/knots2016.html`. Not all the material in these notes could be treated during the talk due to time constraints. The superfluous material has been kept in for the interested reader.

These notes take heavily after some lecture notes by Lackenby [Lac00].

The author thanks Roland van der Veen for his help with the material.

The knots that we will encounter today are the so-called hyperbolic knots. They are those for which their complement (in S^3) has a complete and finite volume hyperbolic structure.

1 The punctured torus

As an introduction to today's topic, let's consider a lower dimensional analogue. We embed a point into the torus $S^1 \times S^1$ and call its complement T the punctured torus. There is an obvious metric on T: by drawing the standard representing square for $S^1 \times S^1$ where the (unique) vertex is the embedded point, and giving this the euclidian metric. The big disadvantage of this is that it is not complete: Cauchy sequences converging to the vertex (in the non-punctured torus) have no limit point in the punctured torus. Alternatively, one may say that geodesics can't always be extended indefinitely as depicted by the red dotted line in Figure 1.

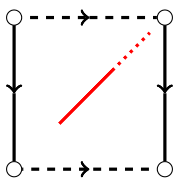

Figure 1: The punctured torus with a non-extendable geodesic.

There is a better metric possible –which is complete– and that one will turn out to be hyperbolic.

2 Hyperbolic manifolds

Definition 2.1. A *Riemannian n-manifold* is a smooth n-manifold M equipped with a metric g which is a family of inner products g_p on the tangent space T_pM which is subject to the requirement that if X and Y are vector fields on M then $p \mapsto g_p(X(p), Y(p))$ is smooth.

For any Riemannian manifold, there exists a measure of 'curvature'. Giving the formal definition would take us too far from today's main topic and is omitted. From the metric one can also define a volume form, giving each Riemannian manifold a possibly infinite volume.

Definition 2.2. *Hyperbolic n-space* is the unique complete simply-connected Riemannian n-manifold with constant curvature -1. It is denoted $\mathbb{H}^n$.

There are several models of $\mathbb{H}^n$ that can be of use. We give two.

- Upper half space. Let U^n denote the set

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}.$$

The metric is given by the formula

$$g_x(u, v) = \frac{\langle u, v \rangle_{\text{Eucl}}}{x_n^2}.$$

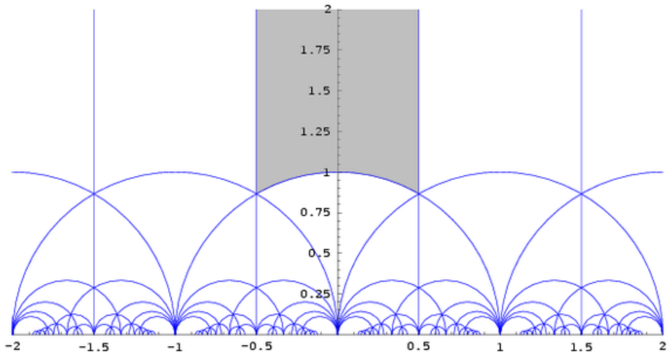

Figure 2: The upper half plane ($n = 2$) model with some hyperplanes. The grey area forms a fundamental domain for the action of the modular group, and is of no importance for our goal today. (picure from `en.wikipedia.org/wiki/Modular_group`)

- Poincaré disc model. Let D^n denote the set

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 < 1\}$$

and assign a metric by the formula

$$g_x(u, v) = \left(\frac{2}{1 - (x_1^2 + \cdots x_n^2)^2} \right)^2 \langle u, v \rangle_{\text{Eucl}}.$$

It is a fact that these two models are isometric and we can translate isometries from one into isometries of others. Hence it makes sense to call both of them a model for $\mathbb{H}^n$. We will switch freely between the models, where these two have the advantage that angles are Euclidian.

In the second model it makes sense to call the bounding set

$$S_\infty^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\}$$

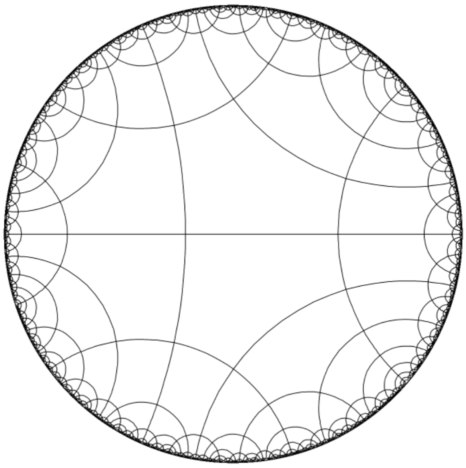

Figure 3: The Poincaré disc ($n = 2$) model with some hyperplanes. (picture from `http://jamesbowthorpe.com/post/93893567397`)

the *sphere at infinity*. This set (and its corresponding sets in the other models) will be useful in our study. In the upper half space model it corresponds to $\mathbb{C} \cup \{\infty\}$.

The second model has a further advantage that it gives a sequence of canonical inclusions $D^1 \subset D^2 \subset \ldots \subset D^n$.

2.1 Isometries

There are two important classes of isometries of hyperbolic space:

(1) These are isometries of the disc model that are restrictions of linear orthogonal maps $\mathbb{R}^n \to \mathbb{R}^n$ that fix the origin.

(2) These are isometries of the upper half space model that are restrictions of maps $\mathbb{R}^n \to \mathbb{R}^n$ of the form $x \mapsto \lambda Ax + b$ with $\lambda \in \mathbb{R}_{>0}$, A an orthogonal map having e_n as eigenspace, and $b \in \mathbb{R}^{n-1} \times \{0\}$.

Theorem 2.3. *The group of isometries* $\mathrm{Isom}(\mathbb{H}^n)$ *is generated by all isometries of classes 1 and 2.*

Proof. This is [Lac00] Corollary 2.2. □

Corollary 2.4. *Let $h : \mathbb{H}^n \to \mathbb{H}^n$ a (hyperbolic) isometry and let k be a positive integer. Then h*

- *h extends to a homeomorphism $S^{n-1}_\infty \to S^{n-1}_\infty$,*
- *preserves the set of codimension-k hyperspaces,*
- *preserves the angles between S^{n-1}_∞ and arcs intersecting S^{n-1}_∞.*

Proof. One can directly check that these statements hold of isometries from classes 1 and 2. □

The extension map in 2.4 is always injective, but that is not a formal consequence of the above theorem.

Definition 2.5. A k-dimensional hyperplane in $\mathbb{H}^n$ is the image of $D^k \subset D^n$ after an isometry. A half-space is the closure in D^n of one component of the complement of a codimension one hyperplane.

In the disc model, hyperplanes are semi-spheres perpendicular to the boundary of the disc and diagonal cross sections. In the upper half space model, hyperplanes are semi-spheres or hyperplanes perpendicular to the boundary hyperplane $x_n = 0$.

We now turn to the isometries of the hyperbolic spaces $\mathbb{H}^3$ and $\mathbb{H}^2$ which are most useful for us.

Definition 2.6. Let $\mathrm{Isom}^+(\mathbb{H}^n)$ denote the subgroup of orientation preserving isometries of $\mathbb{H}^n$.

Definition 2.7. For a field k, define the group $\mathrm{PSL}_2(k) = \mathrm{SL}_2(k)/\{\mathrm{id}, -\mathrm{id}\}$.

Any element $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$ corresponds uniquely to a Möbius map $z \mapsto \frac{az+b}{cz+d}$, which is a homeomorphism of $\mathbb{C} \cup \{\infty\}$.

Theorem 2.8. $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$

Proof. This is [Lac00] Theorem 5.1.

The proof is essentially proving that the image of the set of Möbius maps and the set of extensions of elements of $\mathrm{Isom}^+(\mathbb{H}^3)$ in $\mathrm{Homeo}(\mathrm{S}^2_\infty)$ coincide. For this, one needs to study the fixed points of isometries. □

Theorem 2.9. $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$

Proof. This is [Lac00] Theorem 5.5. It is also well-known for people who know about modular forms. □

2.2 Mostov Rigidity

Just for 'ordinary' manifolds, we can define hyperbolic manifolds using charts to hyperbolic space where gluing maps are supposed to be isometries.

Definition 2.10. We call a manifold *closed* if it is compact and has empty boundary.

A lot of information of hyperbolic manifolds is contained in their fundamental group, as made explicit in the following theorem.

Theorem 2.11 (Mostov Rigidity). *Let M and N be either a complete and finite volume or a closed hyperbolic n-manifolds, with $n \geq 3$. If $\pi_1(M)$ and $\pi_1(N)$ are isomorphic, then M and N are isomorphic hyperbolic manifolds.*

Proof. See [Lac00], page 6. □

Remember that 'complete and finite volume hyperbolic structure' is precisely the property we imposed on the knots of today's interest.

2.3 The punctured torus again

We consider the following ideal polyhedron in the upper half plane:

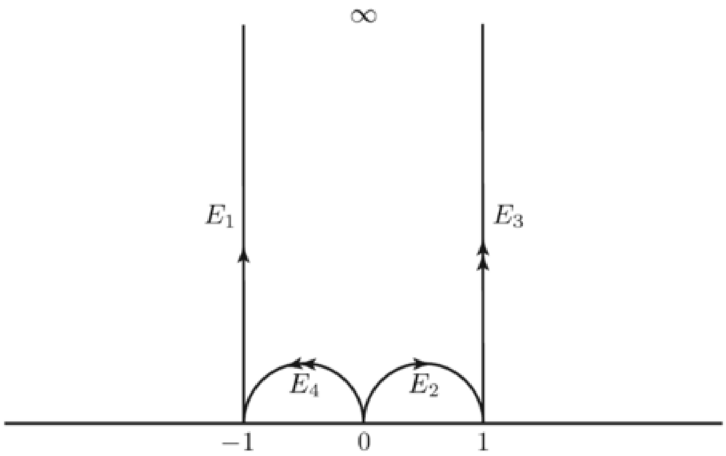

Figure 4: The punctured torus as an ideal square. The metric here is complete: sequences that were Cauchy in the euclidian metric which would converge to the vertex are no longer Cauchy.

The gluing of the sides is done using

$$\begin{aligned}\phi_1 : E_1 &\to E_2,\\ z &\mapsto \frac{z+1}{z+2};\\ \phi_3 : E_3 &\to E_4,\\ z &\mapsto \frac{z-1}{-z+2}\end{aligned}$$

and $\phi_2 = \phi_1^{-1}$ and $\phi_4 = \phi_3^{-1}$.

3 The figure-8 knot

In this section, we will study the hyperbolic structure on the complement of the figure-8 knot in S^3. All of the pictures in this and section the next come from [Lac00] chapter 7.

We'll glue two tetrahedra together in the way depicted in Figure 6. In the quotient space M we glue all the edge into two sets of three and the vertices are all glued to a single point v.

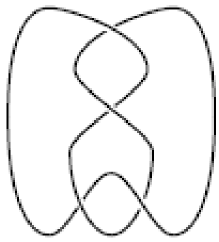

Figure 5: For those knot in the now: the figure-8 knot.

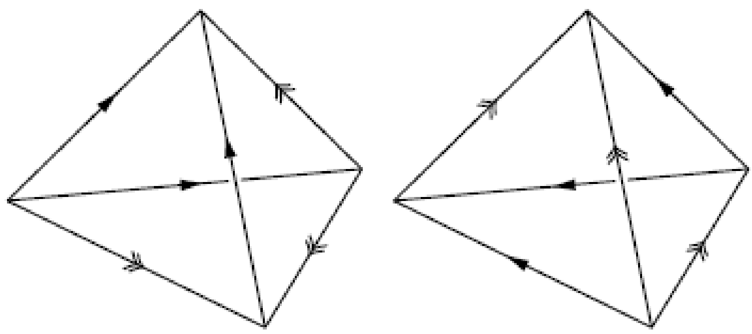

Figure 6: Gluing the two tetrahedra as depicted gives us the quotient space M with all the vertices glued to a single point v.

The topological space M is not a manifold since no small neighbourhood of v is simply-connected. However, $M \setminus v$ is a 3-manifold, and one that is of interest, as shown in the following theorem.

THEOREM 3.1. *As topological spaces $M \setminus v$ is homeomorphic to $S^3 \setminus L$, where L is the figure-8 knot.*

Proof. We start out by making a simplex out of the knot and its complement. We put four 0-cells on the knot and use four 1-cells to cover the knot as in Figure 7. We add two more 1-cells (numbered 1 and 2) to aid us in the proof. We also give each 1-cell an orientation in order to more easily keep track of them.

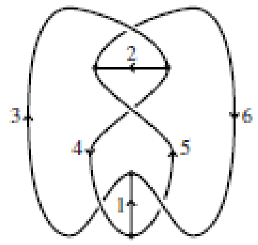

Figure 7: The 0-cells and 1-cells of the simplex. Cells 3 to 6 form the knot, cells 1 and 2 are added for the proof.

Now we add four 2-cells (named A to D) as in Figure 8. Remember that we're doing all of this embedded in S^3, so we can indeed have the 2-cells non-intersecting.

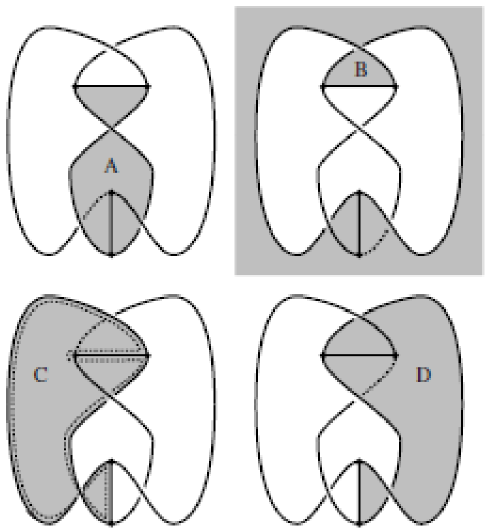

Figure 8: The 2-cells A to D are added. The dotted line for cell C is a guide to the eye in order to make sense of the attachment to the 1-cells. Some areas of different 2-cells seem to overlap, but embedded in S^3 we can make them non-intersecting (as is expected of a cell decomposition).

We've added the 1-cells 1 and 2 in order to deform this cell decomposition such that its complement stays homeomorphic. For this we thicken the 1-cells 1 and 2 and 'pull apart' the knot through them. Afterwards we shrink the blown-up 1-cells back into 1-cells. This process is visualized in Figure 9. Remark that during the process, the space of made up by the 0-cells and 1-cells does not stay homeomorphic, but its complement does and that is the space we care about. After this process, we arrive at the cell decomposition in Figure 10. During the process the 2-cells have been deformed homeomorphically and we can find them in the new picture. Cell A is the middle one, cell B is still the outer one, C the left-hand one and D the right-hand one.

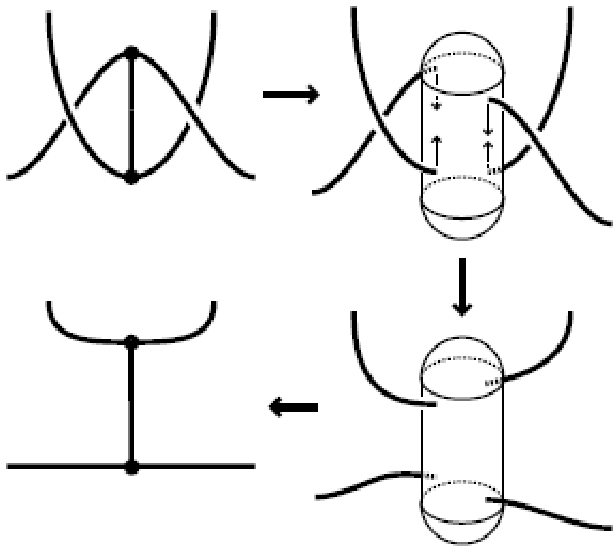

Figure 9: This figure tries to make clear how the deformation process near cell 1 behaves.

It is clear that the decomposition of 0, 1, and 2-cells can be made in a plane. Thus in order to make up for the complement in S^3, we need to add two 3-cells and glue them to the cell decomposition from Figure 10. Indeed, these both have four faces and they are deformed tetrahedra. Let's draw them in Figure 11 and add the labels back in. Remark that when walking around the areas corresponding to the 2-cells, we sometimes consecutively walk along cells 1 and 2 in two opposing directions.

Now we are ready to finish the proof. Notice that each line in Figure 11 contains either the 1-cell 1 or the 1-cell 2. We retract each 1-cell numbered 3 to 6 (those are precisely the ones that made up the figure-8 knot) to the neighbouring 0-cell. Doing so, we arrive at two tetrahedra with the same gluing data as found in Figure 6. After having removed the vertices we arrive at the conclusion of the theorem. □

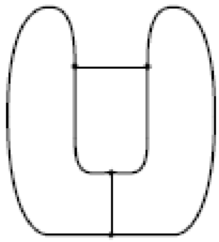

Figure 10: The cell decomposition of the knot after the homeomorphism pulling it apart. The labels have been removed, but it is easy to see that cell A is the middle 2-cell, B the outer one, C the left-hand one and D the right-hand one. Had we kept track of the orientation of the 1-cells in this figure, then the orientation of cells 1 and 2 (which are both still in their original place) would have been reversed in the deformation process.

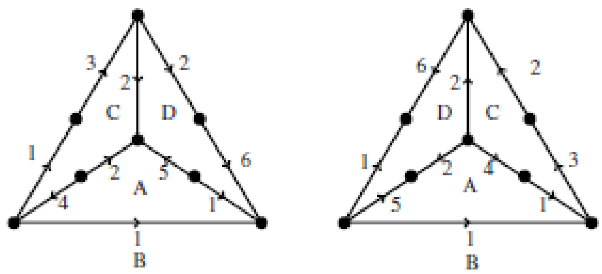

Figure 11: The two 3-cells with corresponding gluing data labelled.

4 Gluing ideal tetrahedra

4.1 Ideal polyhedra

It is convenient to use nice pieces of hyperbolic space to glue together a hyperbolic manifold. These will be ideal polyhedra.

Definition 4.1. A *polyhedron* in $\mathbb{H}^n$ is a compact subset that is the intersection of a finite collection of half-spaces. An *ideal polyhedron* is the intersection of a finite number of half-spaces, whose closure in $\mathbb{H}^n \cap S^{n-1}_\infty$ intersects S^{n-1}_∞ in a finite number of points, and which further has no vertices in $\mathbb{H}^n$. An *ideal tetrahedron* is an ideal polyhedron defined by 4 points on S^2_∞.

Fact 4.2. In order to ensure that we can suitably glue together some ideal tetrahedra, we need only check that all points on the gluing interface, the interior angles sum to 2π.

In section 3, we have used tetrahedra with their vertices removed to glue together the complement of the figure-8 knot. By doing so, we can give this complement a hyperbolic structure, by interpreting these tetrahedra as ideal tetrahedra.

Fact 4.3. Ideal tetrahedra are determined by choosing their vertices in $\mathbb{C} \cap \{\infty\}$ to be at the points 0, 1, ∞, and some $z \in \mathbb{C} \setminus \mathbb{R}$ with $\mathrm{Im}(z) > 0$. There is some choice involved in picking z, which corresponds to the choice of which vertices to place at 0, 1 and ∞. A simple calculation shows that, given that we want to preserve the orientation, the choice is between z, $\frac{1}{1-z}$ and $1 - \frac{1}{z}$.

Lemma 4.4. *The interior angles between any three faces of an ideal tetrahedron sum to π.*

Proof. We work in the upper half space U^3 and assume the hyperplanes containing these faces to be vertical Euclidian planes. (We may do so by placing their common vertex at ∞ and applying an isometry.) Since in this model angles are Euclidian, and the triangle defined by the three faces is now also Euclidian, the lemma follows. See Figure 12 for a visualization.

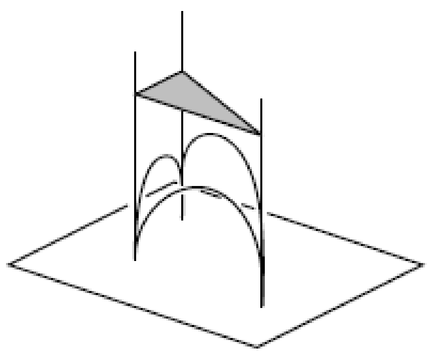

Figure 12: Three faces of an ideal tetrahedron.

$\square$

Let M be a topological space obtained by having glued together a finite number of ideal tetrahedra. Each of these tetrahedra have a hyperbolic structure, but at the faces this structure a priori need not continue. The following theorem makes sense of the fact pointed out above that the only thing that needs to be checked is that the angles match up to 2π.

At each edge i, let $w_{i1}, \ldots, w_{1k}$ be the complex numbers defining the k tetrahedra glued along edge i. We must make a coherent choice for these and we do so by choosing w_{ij} from z, $\frac{1}{1-z}$, $1-\frac{1}{z}$ in such a way that the interior angle of the triangle at the origin is the same as the interior angle at the edge i.

THEOREM 4.5. *Let M be as above with the condition that ∂M is a collection of tori. Then $M \setminus \partial M$ inherits a hyperbolic structure if and only if for each edge i the formula $\prod_j w_{ij} = 1$ holds.*

Proof. This is [Lac00] Theorem 8.4. $\square$

4.2 The figure-8 knot again

For the complement of the figure-8 knot we arrive at two equations, one for each of the edges of the two tetrahedra.

$$1 = z_2 z_1 (1 - \frac{1}{z_2}) z_1 z_2 (1 - \frac{1}{z_1})$$

and

$$1 = \left(\frac{1}{1-z_2}\right)\left(1-\frac{1}{z_1}\right)\left(\frac{1}{1-z_2}\right)\left(\frac{1}{1-z_1}\right)\left(1-\frac{1}{z_2}\right)\left(\frac{1}{1-z_1}\right)$$

These two equations are in fact equivalent, and also equivalent to the equation

$$z_1 z_2 (1 - z_1)(1 - z_2) = 1. \tag{1}$$

In section 3, we glued the tetrahedra together in such a way that they need to be regular. Solving (1) with $z_1 = z_2$, we find $z = \zeta_6$, a choice of primitive sixth root of unity.

The number field defined by adjoining ζ_6 is $\mathbb{Q}(\zeta_6)$ and equals $\mathbb{Q}(\sqrt{-3})$. This number field will come up in the next section.

5 Hyperbolic structures

In today's last section, we make a link between today's and next week's topic. Most of this section is from [Lac00] chapters 9 and 10, and [Rat94] chapter 8.4.

Let M be a Riemannian manifold and $\tilde{M}$ be its universal cover. Then $\tilde{M}$ inherits a metric, covering transformations are isometries, and if M is complete then so is $\tilde{M}$.

LEMMA 5.1. *Let M and N be hyperbolic manifolds with local isometries $f_1, f_2 : M \to N$. Suppose that M is connected and that f_1 and f_2 agree on some non-empty open subset of M. Then $f_1 = f_2$.*

Proof. Lackenby Lemma 10.2. The proof is basically proving that the set on which f_1 and f_2 agree is closed as well as open and uses that isometries act rigidly, which is the property that agreeing on a non-empty open subset implies being equal. The technicalities are standard. □

Let M be a hyperbolic manifold and let $p : \tilde{M} \to M$ be its universal cover with inherited structure of a hyperbolic manifold. Then each covering transformation $\tau : \tilde{M} \to \tilde{M}$ is an isometry. Pick a basepoint $x_0 \in \tilde{M}$ and a connected chart $\phi : U_0 \to \mathbb{H}^n$ around it. Then ϕ can be extended to a local isometry $D : \tilde{M} \to \mathbb{H}^n$ which is called the *developing map*.

We can give $\phi \circ \tau$ as a chart around $\tau^{-1}(x_0)$ and thereby define D in a neighbourhood of $\tau^{-1}(x_0)$. Since gluing maps between charts are supposed to be isometries of $\mathbb{H}^n$, there exists an element $g_\tau \in \mathrm{Isom}(\mathbb{H}^n)$ such that

$$D|_{\tau^{-1}(U_0)} = g_\tau^{-1} \circ \phi \circ \tau$$

holds, making the following diagram commute

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tau"] \arrow[d, "D"] & \tilde{M} \arrow[d, "D"] \\ \mathbb{H}^n \arrow[r, "g_\tau"] & \mathbb{H}^n \end{tikzcd}$$

By Lemma 5.1, this diagram commutes on all of M and by pasting commutative diagrams together, we see that $g_{\sigma\tau} = g_\sigma g_\tau$ holds and hence we have a group homomorphism

$$\eta : \mathrm{Aut}(p) \to \mathrm{Isom}(\mathbb{H}^n).$$

REMARK 5.2. The developing map D and the *holonomy map* η are not unique. The developing map is unique up to composition with an element of $\mathrm{Isom}(\mathbb{H}^n)$ and the holonomy map is unique up to conjugation of an element of $\mathrm{Isom}(\mathbb{H}^n)$.

FACT 5.3. For orientable manifolds, the holonomy η has image in $\mathrm{Isom}^+(\mathbb{H}^n)$.

REMARK 5.4. Remember that the group of covering transformations of a universal cover of a space X is isomorphic to $\pi_1(X)$. Further remember that we have already seen that $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$. We have therefore found a map $\eta : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.

In the case where M is complete, η is injective and $M = \mathbb{H}^n/\eta(\pi_1(M))$ holds.

Remember that η was defined up to choice of x_0 and U_0. Therefore different choices give different hyperbolic structures to M, and different maps into $\mathrm{PSL}_2(\mathbb{C})$.

THEOREM 5.5. *Let M be a hyperbolic n-manifold. Then the set of hyperbolic structures on M is in bijective correspondence to the set of conjugacy classes of homomorphisms $\pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$.*

For every such (conjugacy class of such) morphism on the right, the traces of that morphism generate a number field. By Mostov Rigidity 2.11, there is a canonical choice for a hyperbolic structure on a hyperbolic manifold, and that is the unique one that is complete. Therefore every hyperbolic knot has a canonical associated number field.

EXAMPLE 5.6. For the figure-8 knot, that number field is $\mathbb{Q}(\sqrt{-3})$, corresponding to equation (1) arising from gluing its complement from two ideal tetrahedra.

Bibliography

[Lac00] Marc Lackenby. Lecture notes for hyperbolic manifolds. available on `http://people.maths.ox.ac.uk/lackenby/`, 2000.

[Rat94] John G. Ratcliffe. *Foundations of Hyperbolic Geometry*. Springer Verlag, 1994.