Mastermath Elliptic Curves, Homework 3
Due: 6th October 2015, 10:15

Students are expected to (try to) solve all problems below. The ones marked as "Homework" are to be handed in and count towards your grade according to the rules on the web page.

All problems and their solutions are part of the course, and could play a role in the exam. More importantly, they help you digest the material of the previous lecture and help you prepare for the next lecture.

Problem 16 (Homework)**.** Let f be a non-constant meromorphic function on $\mathbb{C}$. A number $\omega \in \mathbb{C}$ is said to be a *period* of f if, for all $z \in \mathbb{C}$, we have $f(z+\omega) = f(z)$. Let Λ be the set of periods of f. Prove that Λ is a discrete subgroup of $\mathbb{C}$, and deduce that Λ is of one of the following three forms:

$$\Lambda = \{0\}; \qquad \Lambda = \mathbb{Z}\omega \quad (\omega \neq 0); \qquad \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \quad (\text{with } \mathbb{C} = \mathbb{R}\omega_1 + \mathbb{R}\omega_1).$$

Problem 17 (Homework)**.** Let $\Lambda \subset \mathbb{C}$ be the lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$. Show that the solutions to $\wp'_\Lambda(z) = 0$ are $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$ and their translates by Λ.

Problem 18. Let $\Lambda \subset \mathbb{C}$ be a lattice. Show that the non-constant meromorphic solutions to the differential equation

$$(y')^2 = 4y^3 - g_2(\Lambda)y - g_3(\Lambda)$$

are the functions $\wp_\Lambda(z - z_0)$ for $z_0 \in \mathbb{C}$. What are the constant solutions?

Problem 19 (Homework)**.** Let $\Lambda \subset \mathbb{C}$ be a lattice, and $\wp = \wp_\Lambda$ the Weierstrass $\wp$-function associated to Λ. Let $z_1, z_2 \in \mathbb{C}$ be two points not in Λ, such that $z_1 + z_2 \notin \Lambda$.

(a) Show that there exist constants $a, b \in \mathbb{C}$ such that the function

$$f(z) = \wp'(z) - a\wp(z) - b$$

satisfies $f(z_1) = f(z_2) = 0$.

(b) Prove that f also satisfies $f(-z_1 - z_2) = 0$.

(c) Use the cubic equation relating $\wp$ and $\wp'$ to deduce the addition formula:

$$\wp(z_1+z_2) + \wp(z_1) + \wp(z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2.$$

(This exercise may feel very familiar.)

Problem 20. Let $G_k = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-k}$ be the Eisenstein series of order k, and define $G_2 = G_1 = 0$ and $G_0 = -1$.

(a) Show that

$$(k-1)(k-2)(k-3)G_k = 6 \sum_{j=0}^{k} (j-1)(k-j-1) G_j G_{k-j}$$

for all $k \geq 6$. [Hint: $\wp'' = 6\wp^2 - 30G_4$.]

(b) Show that $G_8 = \frac{3}{7}G_4^2$, $G_{10} = \frac{5}{11}G_4G_6$ and $G_{12} = \frac{25}{143}G_6^2 + \frac{18}{143}G_4^3$ and that, more generally, every Eisenstein series can be computed recursively from G_4 and G_6 by the formula

$$(k^2-1)(k-6)G_k = 6 \sum_{j=4}^{k-4} (j-1)(k-j-1) G_j G_{k-j}.$$

Problem 21 (Homework)**.** Let $L(n0)$ be the vector space of all meromorphic functions on the torus $T = \mathbb{C}/\Lambda$ having a pole of order at most n at 0, and no other poles. Prove:

$$\dim_{\mathbb{C}} L(n0) = \begin{cases} n & n > 0; \\ 1 & n = 0. \end{cases}$$