A smooth introduction to knots

Roland van der Veen

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The purpose of these knotes is to give a smooth account of some basic notions in knot theory. We briefly treat the genus, framing, connected sum, satellites the prime decomposition for knots.

1 knots

There are several definitions of what a knot is and what equivalence of knots means. To avoid wild topology and clunky piecewise things we work in the C^∞ or smooth setting. Knots are generally a codimension 2 phenomenon but here we restrict to embeddings of circles into three dimensional space. Instead of $\mathbb{R}^3$ it is almost the same but often much more convenient to consider our knots as sitting inside the (oriented) 3-sphere $\mathbb{S}^3$. It has the advantage of being compact and much more symmetric. Many definitions also work if one replaces the three-sphere by other 3-manifolds.

Definition 1. *A knot K is a smooth embedding $K : \mathbb{S}^1 \to \mathbb{S}^3$. Two knots K, K' said to be equivalent if there is an orientation preserving diffeomorphism $\phi : \mathbb{S}^3 \to \mathbb{S}^3$ such that $\phi(K) = K'$.*

We will not be very careful in distinguishing the knot from its equivalence class. We also sometimes identify the knot with its image.

Nice examples are the *torus knots*. For any pair of coprime integers (p, q) the p, q-torus knot $T(p, q) : \mathbb{S}^1 \to \mathbb{S}^3$ as defined as follows. View the 3-sphere as the sphere in $\mathbb{R}^4 = \mathbb{C}^2$ with radius $\sqrt{2}$ and also $\mathbb{S}^1 \subset \mathbb{C}$ as the unit circle. For $t \in \mathbb{S}^1$ we set

$$T(p, q)(t) = (t^p, t^q)$$

Using stereographic projection or some similar smooth map $\mathbb{S}^3 \to \mathbb{R}^3$ we may project the torus knots down to $\mathbb{R}^3$ where they can be seen to wrap around on the surface of a torus. p-times through the hole, q-times around the hole. One of the benefits of working in $\mathbb{S}^3$ as opposed to $\mathbb{R}^3$ is that in the former it's not hard to see that $T(p, q)$ is equivalent to $T(q, p)$, in $\mathbb{R}^3$ this is a little more challenging. The knot $T(2, 3)$ is called the (righthand) trefoil. In many ways it is the simplest possible non-trivial knot. Its mirror image $T(-2, 3)$ is non-equivalent and is called the left-hand trefoil.

Problem 1. *Given a plane curve* $P = \{f(x,y) = 0 | x, y \in \mathbb{C}\} \subset \mathbb{C}^2$ *and* $f \in \mathbb{C}[x,y]$ *with a singularity at* $(0,0)$ *notice that for sufficiently small* $\epsilon > 0$ *the set* $P \cap \epsilon\mathbb{S}^3$ *is actually a knot inside* $\epsilon\mathbb{S}^3$*. For* $f(x,y) = x^p - y^q$ *one gets a torus knot. What happens for more general singular* f*? Say* $x^5 - y^2 + 4xy^3$*?*

Certain aspects of (equivalence classes of) knots are traditionally called *knot invariants*. They can be of use in deciding whether or not two knots are in fact equivalent but that is not the main point. Rather the study of knots and their invariants should be viewed as a convenient setting for seeing many different kinds of mathematics interact. Hyperbolic geometry, number theory and representation theory to name a few of my personal favourites.

2 Knot as the boundary of a surface

One way to look at knots is to see them as the boundary of something. First if our knot (or rather its image) K bounds a disk D embedded in $\mathbb{S}^3$ then we say K is trivial. The reader should check that all trivial knots are in fact equivalent, so we may call them collectively the *unknot*. What about more general knots like our torus knots? There are two ways to proceed: 1) See if our knot happens to bound a surface of higher genus in $\mathbb{S}^3$ and 2) See if it bounds some disk in $\mathbb{R}^4$. We will mostly focus on the first question but will make some comments on 2) at the end of the section.

2.1 Genus of a knot

By a surface we will mean a compact, orientable smooth 2-manifold, possibly with boundary. Such surfaces are classified by two numbers: their genus g and the number of boundary components b (all circles). The Euler characteristic is $\chi = 2 - 2g - b$ and is usually more convenient for computations. The boundary of a surface Σ is denoted $\partial\Sigma$.

Definition 2. *An embedded surface* $\Sigma \subset \mathbb{S}^3$ *is called a Seifert surface for a knot* K *if* $\partial\Sigma = K$*. The genus of a knot is the minimal genus of a Seifert surface of the knot.*

By definition the unknot is the only knot with genus 0. The genus of the torus knots is known to be $genus(T(p,q)) = \frac{(p-1)(q-1)}{2})$. Giving a Seifert surface that has genus equal to this number should be doable for the reader. At least for the trefoil. Proving that there are no Seifert surfaces for that torus knot with smaller genus is more difficult. For the trefoil this amounts to proving that it is not equivalent to the unknot.

It should be noted that there are more knots than just torus knots, there are infinitely many knots with genus equal to 1. Simple examples of such are the twist knots.

Lemma 1. *Any knot has a Seifert surface.*

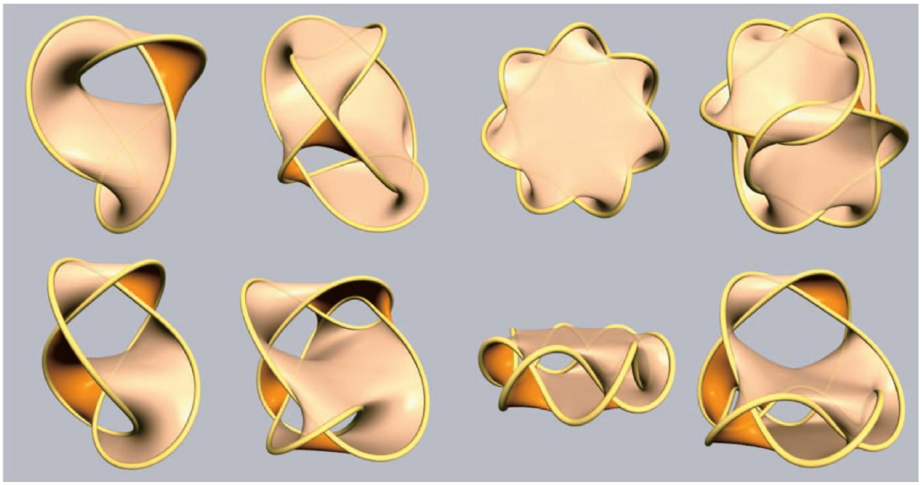

Figure 1: Seifert surfaces for some simple knots. Drawn using the program SeifertView by van Wijk. (some knots are equivalent, do you see which?)

Proof. (Sketch)
Choose a knot diagram and orient the knot. Every crossing locally divides the knot diagram into four regions. Shade the region that is on the left of you when travelling along either of the two parts of the knot. Do the same for the opposite region. Now construct a surface by taking a disk for each shaded region and connect the disks by half twisted bands as indicated by the crossings. By construction this surface has the knot as its boundary. It is also orientable and hence a Seifert surface. □

Although this proof is constructive it is not great since choosing a knot diagram (like choosing a basis in linear algebra) is usually a sign of weakness. Not every Seifert surface comes from the above construction. It was shown that all Seifert surfaces constructed from a diagram of the knot 10_{165} have genus 3 while the knot actually has genus 2!

The Alexander polynomial $\Delta_K(t)$ gives a nice lower bound for the genus: the difference between the highest and the lowest exponent of t in the polynomial must be less than or equal to twice the genus. The more sophisticated Floer homology which is a generalization of the Alexander polynomial actually determines the genus on the nose.

Our Seifert surfaces can also be employed to construct interesting covering spaces of the knot complement, such as the infinitely cyclic one leading to the Alexander polynomial. We will not pursue this here but give a small illustration of the linking number instead.

Definition 3. *The linking number of two disjoint knots $K, L \subset \mathbb{S}^3$ is defined to be the oriented intersection number of L with a Seifert surface of K.*

The reader is invited to see that this definition does not depend on the choice of Seifert surface and is actually symmetric in K and L.

2.2 Slice knots

Apart from considering 3-dimensional things we may also ask what happens if we view our knot as sitting inside $\mathbb{S}^3 \subset \mathbb{R}^4$. Any knot bounds a topologically embedded disk in $\mathbb{R}^4$ but only some knots (the *slice knots*) bound smoothly embedded disks in $\mathbb{R}^4$. As usual with 4-dimensional matters this quickly leads to an open problem:

Problem 2. *(Slice-Ribbon conjecture, 1962)*
Say a ribbon disk is a disk in $\mathbb{R}^3$ that self intersects along finitely many disjoint intervals, such that the preimage of each interval consists of two intervals: one interior and one starting and ending on the boundary. Are all slice knots ribbon knots?

The amusing thing about this is that all experts believe the conjecture is false, potential counter-examples such as the knot shown below have been proposed but nobody can prove anything.

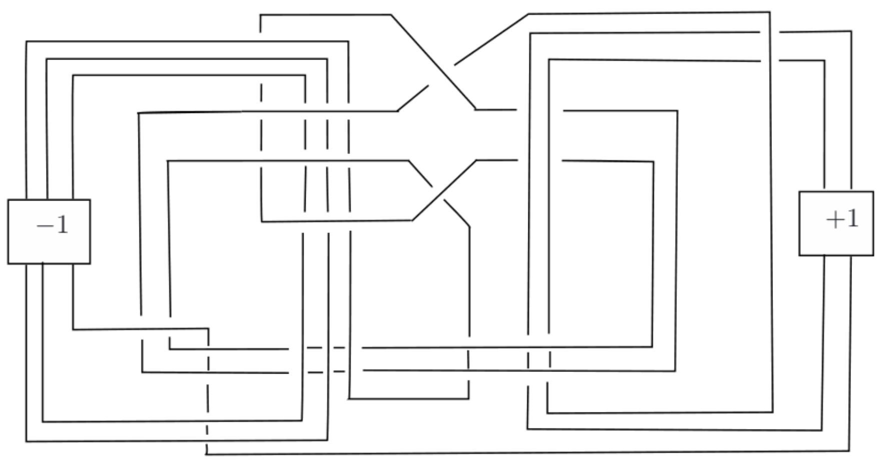

Figure 2: A knot that is proven to be slice but may not be ribbon. The boxes $+1$ and -1 indicate that the strands get twisted around by one full twist inside the box.

A theorem by Fox and Milnor says that the Alexander polynomial $\Delta_K(t)$ of a slice knot K has the form $f(t)f(t^{-1})$ for some polynomial f. One reason for trying to generalize the Alexander polynomial so that it can distinguish ribbon knots from slice knots by something like Fox-Milnor.

3 Operations on knots

To start our operations on knots it is useful to define a slightly more general notion of knots called framed knots. They may also be viewed as embedded annuli with a distinguised side.

Definition 4. *A framed knot F is a smooth embedding $F : \mathbb{S}^1 \times [-1, 1] \to \mathbb{S}^3$. The framing (number) of F is the linking number of its boundary components.*

There is actually a unique way to upgrade a knot K to a framed knot F by setting $K(t) = F(t, -1)$ and requiring the two boundary components to have linking number 0. Exercise: draw a framed trefoil knot with framing 0. For notational purposes the blackboard framing is often used. This means that one draws an ordinary knot diagram and agrees that the framing is always parallel to the blackboard. Using the framing we can easily state a fun problem that sets the tone for the remaining section.

Problem 3. *A magic framed knot is a framed knot in $\mathbb{R}^3$ such that its two boundary components can be separated by an embedded 2-sphere. Are there any magic knots?*

The framing also allows us to uniquely specify surgery to produce compact 3-manifolds but this is something for another lecture.

A framed knot can be extended canonically to an embedding of $\tilde{F} : \mathbb{S}^1 \times D \to \mathbb{S}^3$, where D is the unit disk in $\mathbb{C}$ by requiring that $\tilde{F}|_{[-1,1]} = F$. So far we have considered only knots inside the 3-sphere or perhaps $\mathbb{R}^3$ but we can equally well consider knots inside other 3-manifolds such as the solid torus $\mathbb{S}^1 \times D$. Together with the above remark this leads to the satellite construction.

Definition 5. *Given a knot $P : \mathbb{S}^1 \to \mathbb{S}^1 \times D$ (called the pattern) and a framed knot C (called the companion) we may consider the knot defined by $K = \tilde{C} \circ P$.*

When P, C are non-trivial and P intersects each $\{t\} \times D$ at least once we say that K is a satellite knot.

A very special case of the satellite construction is when P intersects some disk $\{t\} \times D$ only once. In that case the framing is unimportant and the construction can be reformulated in terms of connected sum. The essential topological fact one should know is that any smooth embedded 2-sphere separates $\mathbb{S}^3$ into two solid balls.

Definition 6. *Given a knot K in $\mathbb{S}^3$ suppose there exists a smooth embedded 2-sphere S that intersects K in precisely two points p, q. If we delete the part of K in the open ball on one side of S and connect p, q smoothly along a simple arc inside S. In this case we say that we have a connected sum $K = K_1 + K_2$.*

The reader should check that the knots are uniquely defined up to order and equivalence. The operation $+$ is in fact also associative. It is also an interesting exercise to see how the given definition agrees with the special case of the satellite construction mentioned above. An important theorem is that genus (like many other knot invariants) behaves well under connected sum:

Theorem 1. *For any knots we have genus$(K + L)$ = genus(K) + genus(L)*

Proof. (Sketch)
It's not hard to see that one can combine two Seifert surfaces to get the inequality $\leq$. To see that we actually have equality, we use the 2-sphere guaranteed by the fact that we have a connected sum to decompose any Seifert surface for $K + L$ into two pieces, one surface for K and one for L. The proof is by induction on the complexity of the intersection of the surface with the 2-sphere much in the same way as the problem of the magic knots. $\square$

The theorem quickly dashes any hopes of finding an inverse for the connected sum operation: If $K + L$ is the unknot then at least one of K, L must have been the unknot because the genus is non-negative. A corollary of this theorem is the existence of a unique prime factorization for knots in the following sense. A non-trivial knot K is called prime if $K = A + B$ implies either A or B is trivial. By construction any knot can be written as a sum of prime knots. This sum has to be finite or at least all but finitely many terms should be trivial because the knot is smooth. For example knots of genus 1 such as the trefoil must be prime. With some additional effort one sees that up to ordering this factorization is unique.

Knot theory would be much easier if there were more local operations like connected sum. One could use the operations to decompose knots and their invariants and would only have to study small parts at the time. The satellite construction is of some use but already too complicated. Knot theory remains highly non-local. However when one generalizes knots to embeddings of trivalent graphs the situation is much better and this leads to Drinfeld's deep theory of associators.

We end this note with a segue to the topic of hyperbolic manifolds.

Theorem 2. *(Thurston) There are three types of knots:*

1. *Torus knots*
2. *Satellite knots*
3. *Hyperbolic knots*

Proof. This is a special case of Thurston's geometrization program, proved in 2003 by Perelman along with the Poincare conjecture. $\square$

We met the torus knots and the satellite knots but what is a hyperbolic knot? Find out next seminar!