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Classical Knot Invariants and Elementary Number Theory

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1. Introduction and Preliminaries

(1) Knots, equivalent knots and knot types

A knot K is a simple closed curve in a 3-space $\mathbb{R}^3$ or a 3-sphere S^3. (Since S^3 is obtained from $\mathbb{R}^3$ by adding one point ∞, there are no essential differences between knots in $\mathbb{R}^3$ and knots in S^3.) Further, in order to avoid unnecessary complications, we may assume that K is a polygon (with many sides). See Fig. 1.1. Both $\mathbb{R}^3$ and S^3 are oriented. A 3-space $\mathbb{R}^3$ (and S^3) is given a right handed orientation. See Fig. 1.2.

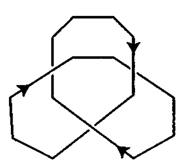

Figure 1.1

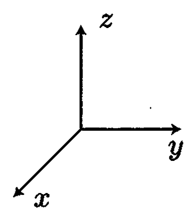

Figure 1.2

The orientation of a knot is usually denoted by an arrow. See Fig. 1.1. Such a knot is called an *oriented* knot. Two knots K_1 and K_2 (in $\mathbb{R}^3$ or S^3) are said to be *equivalent* (or *ambient isotopic*) if there is a homeomorphism f of $\mathbb{R}^3$ (or S^3) onto itself that maps K_1 to K_2, where f preserves orientations of $\mathbb{R}^3$ (or S^3). If K_1 and K_2 are oriented, then f sends K_1 to K_2 including orientations. Intuitively speaking, two knots are *equivalent* if we can move one knot to another knot in $\mathbb{R}^3$ (or S^3) without allowing any self-intersections. See Fig. 1.3. The simplest knot is a knot equivalent to a circle on the plane in $\mathbb{R}^3$. It is called a *trivial knot* or a *unknotted knot.* Fig. 1.4.

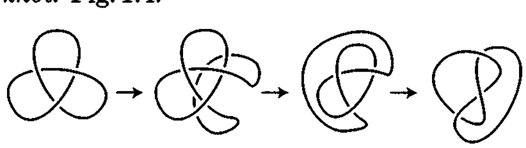

Figure 1.3

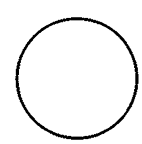

Figure 1.4

Two equivalent knots are said to be of the *same type.*

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(2) Invertibility

Each knot can have two different orientations, one is opposite to another. See Fig. 1.5.

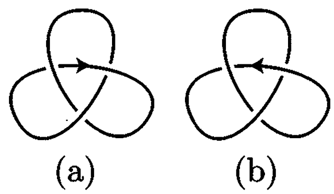


Figure 1.5

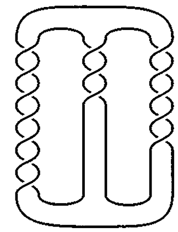


Figure 1.6

It is conceivable that two oriented knots with opposite orientations given to the same knot may not be equivalent. But it was 1964 when H.Trotter first proved that the knot in Fig. 1.6 is one of such knots [T3]. A knot is called *invertible* if two oriented knots with opposite orientations obtained from the same knot are equivalent. Many knots are invertible, but non-invertible knots are by no means rare.

(3) Amphicheirality

Given a knot K in $\mathbb{R}^3$, we can construct a mirror image of K. More precisely, let f be a reflection of $\mathbb{R}^3$, i.e. $f(x,y,z) = (x,y,-z)$. The image of K under f is called the *mirror* image of K. If K is equivalent to its mirror image K^*, then K is said to be *amphicheiral.* If an oriented knot K and K^* with the orientation induced from that of K are equivalent, we say an (oriented) knot K is +amphicheiral. However, if K and K^* with opposite orientation are equivalent, K is called −amphicheiral. For example, the knot in Fig. 1.1 is not amphicheiral, but the proof is rather complicated. The simplest non-trivial amphicheiral knot is shown in Fig. 1.7.

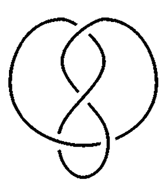


Figure 1.7

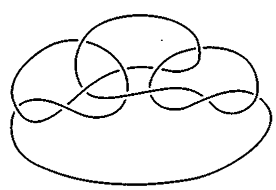


Figure 1.8

(4) Knot invariant

One of the fundamental problems in knot theory is a so-called the *classification problem.* The problem is to construct a list $\mathcal{L}$ of knots for which (1) any knot is equivalent to a knot in $\mathcal{L}$ and (2) two knots in $\mathcal{L}$ are not equivalent. The classification problem has been solved for certain families of knots which we discuss later. However, so far, there are no algorithms by which we can determine whether or not given two knots are equivalent. One of the effective methods to distinguish two knots is to compare their knot invariants. A quantity $\rho(K)$ assigned to a knot K is called a *knot invariant* if $\rho(K_1) = \rho(K_2)$ for equivalent knots K_1 and K_2. One of the earliest knot invariants is probably the Minkowski unit $C_p(K)$ for a knot K and an odd prime p. (See Section 6.) Later, Alexander [A2] introduced the most important knot invariant in knot theory called the *Alexander polynomial.*

The Alexander polynomial is an integer polynomial for any knot, and in particular, for a trivial knot, its value is 1. The Alexander polynomial is one of the major tools to distinguish two knots, however it is not a *complete invariant.* In fact, Kinoshita-Terasaka knot (or KT knot) depicted in Fig. 1.8 is not a trivial knot, but the Alexander polynomial is 1.

(5) **Links**

So far, we have only looked at knots, but many concepts can be extended to a link, a collection of knots. A *link* is a finite ordered collection of mutually disjoint knots $L = K_1 \cup K_2 \cup \cdots \cup K_n$, in $\mathbb{R}^3$ or S^3. Each knot K_i is called a *component* of L. If L consists of n knots, L is called an *n-component link.* If each component is oriented, L is called an oriented link. Two links $L = \{K_1, K_2, \ldots, K_n\}$ and $L' = \{K_1', K_2', \ldots, K_m'\}$ are *equivalent* if (1) $n = m$, and (2) there is a homeomorphism f that preserves the orientation of $\mathbb{R}^3$ (or S^3) and f maps K_1 to K_1', K_2 to $K_2', \ldots, K_n$ to K_n' (since $m = n$). Strictly speaking, the equivalence of links should also be related to how we order the components. However, such a stringent condition is not necessary, since we may suitably reorder the components. Therefore (2) is usually replaced by the following (2A): (2A) There is a homeomorphism f that preserves the orientation of $\mathbb{R}^3$ (or S^3) and maps $K_1 \cup K_2 \cup \cdots \cup K_n$ to $K_1' \cup K_2' \cup \cdots \cup K_m'$. For a link, invertibility or amphicheirality is not a major problem. In fact, the Alexander polynomial is defined only for an oriented link, and if one component K_i reverses its orientation, the Alexander polynomial changes entirely. For a knot, however, the Alexander polynomial does not depend on its orientation. Therefore, the non-invertibility of a knot cannot be detected by the Alexander polynomial. Although these notable differences exist between knots and links, many properties of knots are extended to those of links in a straightforward manner. However, in this article, knots and links are carefully distinguished. Therefore, a link always means a link with more than one component.

2. Knot (or link) Group

(1) **Diagram**

We usually represent a knot K by a graph D with "broken edges" on a plane. See Fig. 1.1. This is called a *regular diagram* of K. A regular diagram gives us a sense of how the knot may in fact lie in 3-dimensions, i.e. it allows us to depict the knot as a spatial diagram on the plane. Since we consider exclusively regular diagrams, we call them simply *diagrams* of knots.

(2) **Knot group**

For a knot K, $\pi_1(\mathbb{R}^3 - K, *)(\simeq \pi_1(S^3 - K, *))$, the fundamental group of the complement of K in $\mathbb{R}^3$ (or S^3), is called the *group of knot* or *knot group* of K, where $*$ denotes a base point, but since two groups with different base points are conjugate, we usually omit $*$. We denote the knot group of K by $G(K)$. $G(K)$ is finitely presented, i.e. $G(K)$ has a presentation in which both the number of generators and relations are finite.

(3) **Wirtinger presentation**

Using a diagram, we can obtain one of finite presentations, called a *Wirtinger presentation,* of $G(K)$. In this presentation, each generator corresponds to an *over-passing* arc, defined below, and each relation corresponds to a crossing point. Let K be an oriented knot and D a diagram. Select one under-crossing point, P_1 say, and trace the arc (according to its orientation) until the next under-crossing point P_2.

See Fig. 2.1. The arc connecting P_1 and P_2 is denoted by s_1, called an *over-passing arc.* The next arc s_2 is the arc connecting P_2 and P_3, and so on. Note that s_2 is a long arc, but s_5 and s_6 are short arcs in Fig. 2.1. A short arc does not pass over any arc. Each arc s_i corresponds to a generator x_i of $G(K)$. See Fig. 2.2.

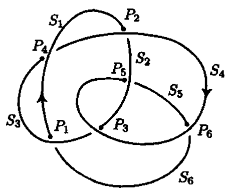


Figure 2.1

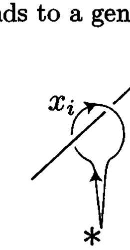


Figure 2.2

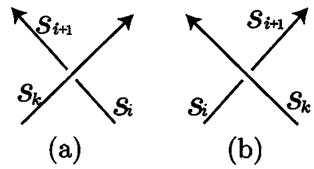


Figure 2.3

x_i is an oriented loop in $\mathbb{R}^3$ (or S^3) that circles once around s_i as is shown in Fig. 2.2. The set of these elements $\{x_1, x_2, \ldots, x_n\}$ is a set of generators of $G(K)$, where n is the number of crossing points in D. A relation of $G(K)$ is obtained as follows: To each crossing point, 3 arcs meet as in Fig. 2.3 (a) or (b). For (a), the corresponding relation is $R_i = x_i x_k x_{i+1}^{-1} x_k^{-1}$. For (b), it is $R_i = x_i x_k^{-1} x_{i+1}^{-1} x_k$.

Therefore, $G(K)$ has a presentation $G(K) = \langle x_1, x_2, \ldots, x_n | R_1 = 1, R_2 = 1, \ldots, R_n = 1\rangle$. It is known that any one of R_i is unnecessary, so that $G(K)$ is presented by n generators and $n-1$ relations, where n is the number of crossing points of D. This presentation of $G(K)$ is called a *Wirtinger presentation* of $G(K)$.

(4) **Examples**

Example 2.1 For the knot with the diagram D in Fig. 2.1,

$G(K) = \langle x_1, x_2, \ldots, x_6 | R_1 = 1, R_2 = 1, \ldots, R_6 = 1\rangle$, where $R_1 = x_1 x_4^{-1} x_2^{-1} x_4$, $R_2 = x_2 x_4 x_3^{-1} x_4^{-1}$, $R_3 = x_3 x_1^{-1} x_4^{-1} x_1$, $R_4 = x_4 x_2 x_5^{-1} x_2^{-1}$, $R_5 = x_5 x_4 x_6^{-1} x_4^{-1}$, $R_6 = x_6 x_3^{-1} x_1^{-1} x_3$.

Example 2.2 A Wirtinger presentation of K in Fig. 2.4 is

$G(K) = \langle x_1, x_2, x_3 | R_1 = 1, R_2 = 1\rangle$, where $R_1 = x_1 x_3 x_2^{-1} x_3^{-1}$, $R_2 = x_2 x_1 x_3^{-1} x_1^{-1}$. We can simplify this presentation as follows. First, $R_2 = 1$ yields $x_1^{-1} x_2 x_1 = x_3$, and hence x_3 can be eliminated. A new presentation has 2 generators x_1 and x_2, and one relation R'_2 that comes from R_1: $R'_1 = x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1} x_1$. If we set $a = x_2 x_1$ and $b = x_2 x_1 x_2$, then a and b generate $G(K)$ and one relation R'_1 becomes $ab^{-2}a^2 = 1$, and thus we have a simple presentation $G(K) = \langle a, b | a^3 = b^2\rangle$.

(5) **Link group**

For an oriented link $L = K_1 \cup K_2 \cup \cdots \cup K_r$, the link group $G(L) = \pi_1(\mathbb{R}^3 - L)$ $(\simeq \pi_1(S^3 - L))$ has a Wirtinger presentation, where the set of generators is $\{x_{i,j} | i = 1, 2, \ldots, r; j = 1, 2, \ldots, m_i\}$, and the set of relations is $\{R_{i,j} = 1 | i = 1, 2, \ldots, r; j = 1, 2, \ldots, m_i\}$, and $R_{i,j}$ is of the form : $x_{i,j} x_{k,\ell}^{e_{i,j}} x_{i,j}^{-1} x_{k,\ell}^{-e_{i,j}}$, where $e_{i,j} = +1$ or-1. $x_{i,j}$ corresponds to an over-passing arc in the ith component K_i of L. As in the case of the knot group, at least one of the relations can be eliminated.

Example 2.3 Let L be a link consisting of n *completely separated* trivial knots. More precisely, L has a diagram consisting of n trivial knots that are separated by $n-1$ parallel lines. See Fig. 2.5 for $n = 3$. Then L is called a *trivial n-component link.*

The group of L is a free group of rank n freely generated by $x_{1,1}, x_{2,1}, \ldots, x_{n,1}$.

Example 2.4 Let L be a 2-component link in Fig. 2.6.

Then $G(L) = \langle x_{1,1} x_{2,1} | R_{1,1} = 1, R_{2,1} = 1\rangle$ where $R_{1,1} = x_{1,1} x_{2,1} x_{1,1}^{-1} x_{2,1}^{-1}$, and

$R_{2,1} = R_{1,1}^{-1}$. Therefore, $G(L)$ is a free abelian group of rank 2. The link L is called a *Hopf link*.

Remark 2.1 The knot (or link) group is an invariant of an *unoriented* knot (or link). Furthermore, the group of a knot (or link) K is isomorphic to that of the mirror image of K. Therefore, we cannot detect the invertibility and amphicheirality of K by its group alone.

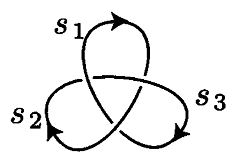


Figure 2.4

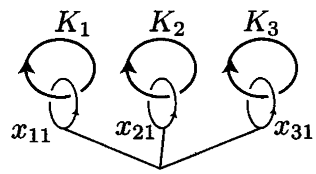


Figure 2.5

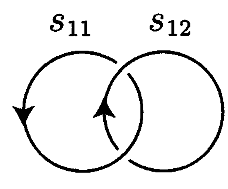


Figure 2.6

3. Braids

(1) **n-braids**

Take a cube B in $\mathbb{R}^3$, and plot n points $A_1, A_2, \ldots, A_n$ on the top of B and other n points $A'_1, A'_2, \ldots, A'_n$ on the base of B. For the sake of neatness, we specify the coordinates of these points. First, describe B as the set $\{(x, y, z) | 0 \le x, y, z \le 1\}$, and define the coordinates of A_i as $(\frac{1}{2}, \frac{i}{n+1}, 1)$ and that of A'_i as $(\frac{1}{2}, \frac{i}{n+1}, 0)$. Now join these $A_1, \ldots, A_n$ to $A'_1, \ldots, A'_n$ by means of n (polygonal) arcs $u_1, u_2, \ldots, u_n$ inside B (except their end points) as follows: (1) u_i and u_j do not intersect if $i \neq j$. (2) u_i joins A_i to A'_k for some k. These arcs u_i are called *strings*. If any plane E parallel to the base of B either intersects each string at one and only one point, or it does not intersect at all, then the set of these n strings in B is called an *n-braid*. Given an n-braid, by projecting the braid onto the yz-plane, we obtain a (regular) diagram of a braid (as in the case of knots) . See Fig. 3.1 (a) or (b).

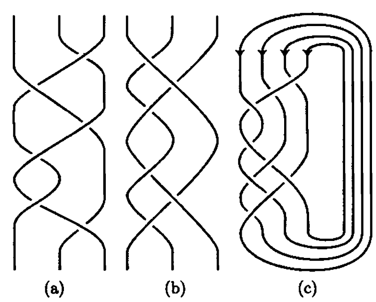


Figure 3.1

As is shown in Fig. 3.1 (b), if each A_i $(i = 1, 2, \ldots, n)$ is connected to A'_i by the ith string u_i, then an n-braid is called a *pure* n-braid. Intuitively, two braids (in a cube) whose end-points we keep fixed, is said to be *equivalent*, if we can continuously deform one to the other without causing any of the strings to intersect each other.

(2) **The n-braid group B_n**

Given two n-braids α and β, we can define the product $\alpha\beta$. First glue the base of the cube that contains α to the top face of the cube that contains β. The gluing together of two cubes produces a rectangular solid in which there exists an n-braid that has been created from α and β. This braid is the *product* $\alpha\beta$. It is well known that under this product, the set of all (equivalent classes of) n-braids forms a group, called the *n-braid group*, denoted by B_n. The group B_n is generated by $n-1$ n-braids, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. See Fig. 3.2.

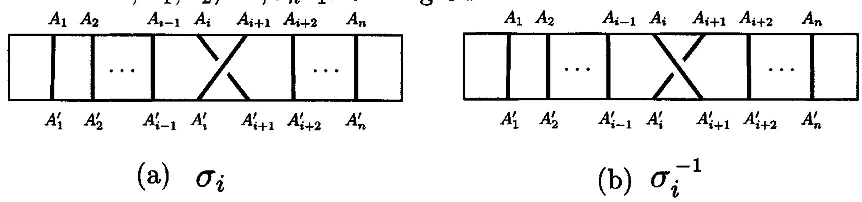


Figure 3.2

There are two types of relations for B_n. One is of the form (I) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|>1$, and another is of the form (II) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. These relations (I) and (II) form a complete set of relations for B_n.

(3) **Examples**

Example 3.1 The 3-braids shown in Fig. 3.1 (a) and (b) are written, respectively, as $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_2^{-1}$ and $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_1$.

Example 3.2 (1) $B_1 = \langle 1 \rangle$ is a trivial group. (2) $B_2 = \langle \sigma_1 | - \rangle$ is an infinite cyclic group.

(3) $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$

(4) $B_4 = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_3 = \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle$

(4) **closed n-braid (or the closure of a braid)**

As is shown in Fig. 3.1 (c), we connect, by a set of parallel arcs that lie outside the square, the points $A_1, A_2, \ldots, A_n$ on the top of the braid α to the points $A_1', A_1', \ldots, A_n'$. The knot or link thus created is called the *closed braid* (or the *closure* of α). We assign to each string an orientation downward. The following theorem due to Alexander connects the braid theory to knot theory.

Theorem 3.1[A1] *Any knot (or link) is equivalent to some closed braid.*

Therefore, to each knot, we can associate an element of some braid group. However, since a knot can be represented as many different closed braids, an element of a braid assigned to a knot is not a knot invariant.

(5) **Braid index**

By Theorem 3.1, we see that any knot (or link) K is a closed n-braid for some $n \geq 1$. If K is a closed m-braid, but K cannot be represented by less number of strings, then m is called the *braid index* of K, denoted by $b(K)$. Obviously, $b(K)$ is an invariant of a knot (or link). It is a hard problem to decide the braid index, and there is no algorithm to determine $b(K)$. If L is an n-component link, then $b(L) \geq n$. For a link L, $b(L)$ depends on the orientation of each component. If L is a closed pure braid, then each component of L is unknotted, but the converse is not true.

4. Special families of knots (I): Torus knots

In the following two sections, we introduce two familiar classes of knots, tours knots and 2-bridge knots. These knots have been studied quite extensively, and, in particular, these knots are completely classified. Now, we begin with an introduction of a new concept.

(1) **Peripheral system**

Let T be a torus in $\mathbb{R}^3$. T may be considered as the boundary of a solid obtained by fatting a knot K in $\mathbb{R}^3$. (Such a solid is called a *tubular neighbourhood* of K.) See

Fig. 4.1 (a). On this torus T, there are two simple (oriented) closed curves, called a *meridian* m and a *longitude* ℓ, see Fig. 4.1. A meridian m is a simple closed curve on T that bounds a disk *inside* T, while ℓ is a simple closed curve on T that bounds an orientable surface *outside* T. (For an existence of a longitude, see Remark 7.1.) Both are oriented as are shown in Fig. 4.1 (a) and (b), where the orientation of ℓ is given so that two oriented simple closed curves ℓ and K are parallel. A pair $\{m, \ell\}$ is called a *peripheral system* of a knot K.

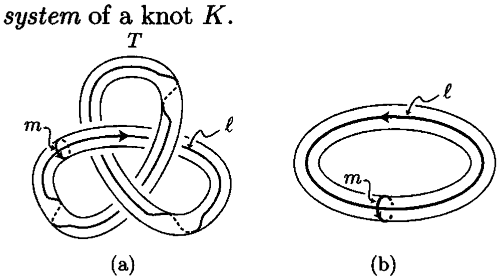


Figure 4.1

Two curves m and ℓ represents two elements of the knot group $G(K)$ in a natural way.

A pair $\{m, \ell\}$ with the knot group $G(K)$ characterizes completely the knot type. More precisely, we have the following theorem due to Waldhausen:

Theorem 4.1 [Wal] *Two oriented knots K_1 and K_2 are equivalent if and only if there exists an isomorphism $f : G(K_1) \longrightarrow G(K_2)$ such that f maps a peripheral system of K_1 to a (conjugate of a) peripheral system of K_2.*

In fact, Dehn's proof of non-amphicheirality of the knot in Fig. 1.1 and also Trotter's proof of the non-invertibility of the knot in Fig. 1.6 are based upon Theorem 4.1.

(2) **Torus knots**

A torus knot is a knot embedded in a *standard torus* T in $\mathbb{R}^3$, where a standard torus means a surface generated by revolving a circle C (on a plane) about a line disjoint to C. Therefore, a standard torus can be considered as the boundary of a tubular neighbourhood of a trivial knot. Let $\{m, \ell\}$ be a peripheral system of a trivial knot K, see Fig. 4.1 (b). A torus knot (or link) of type (p, q), denoted by $T(p, q)$, is an oriented knot (or link) such that (i) $T(p, q)$ crosses a meridian m at exactly $|p|$ points from right to left if $p > 0$ (or from left to right if $p < 0$), (ii) $T(p, q)$ crosses a longitude ℓ at exactly $|q|$ points from right to left if $q > 0$ (or from left to right if $q < 0$). Any (non-trivial) knot on T can be deformed on T to $T(p, q)$ for some integers p and q, and $\gcd(p, q) = 1$. See Fig. 4.2.

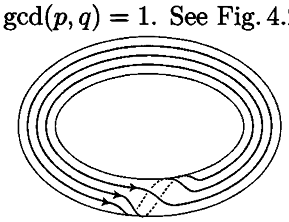


Figure 4.2: $T(3, -2)$

In particular, a meridian m may be considered as a torus knot of type $(0, -1)$ and ℓ as $T(1, 0)$. The number of components of $T(p, q)$ is given by $\gcd(p, q)$. Furthermore, if $p > 0$, then $T(p, q)$ is represented as a closed p-braid $(\sigma_1 \sigma_2 \cdots \sigma_{p-1})^{-q}$. For $p < 0$, see Theorem 4.4. These torus knots have been completely classified.

Theorem 4.2 *A torus knot $T(p,q)$ is a trivial knot if and only if $pq = 0$, or p or q is ± 1.*

Theorem 4.3 *Any torus knot is invertible, and any non-trivial torus knot is not amphicheiral.*

A proof of the second statement will be outlined later.

Theorem 4.4 *Assume that none of p, q, p', q' is 0 or ± 1, and $gcd(p,q) = gcd(p',q') = 1$. Then, (i) $T(p,q) = T(q,p)$ (ii) $T(p,q) = T(p',q')$ if and only if $\{p,q\} = \{p',q'\}$.*

Proof. (i) $T(p,q)$ can be deformed into $T(q,p)$ in $\mathbb{R}^3$ (not on T). A proof of (ii) will be given later.

Theorem 4.5 *The braid index of a non-trivial torus knot $T(p,q)$ is equal to $min\{|p|,|q|\}$.*

5. Special families of knots (II) 2-bridge knots (or rational knots)

(1) **Bridge index**

Any knot (or link) has a diagram with only finitely many local maximal points. The minimal number of local maximal points among all diagrams K can have is called the *bridge index* of K, denoted by $br(K)$. $br(K)$ is an invariant of a knot (or link) K. The bridge index of an n-component link is at least n. Since a knot with the bridge index 1 is a trivial knot, the class of 2-bridge knots is the simplest class of non-trivial knots with respect to the bridge index. It is not easy to determine the bridge index for a knot or link, and there is no algorithm to decide the bridge index.

(2) **2-bridge knot (or link)**

A knot (or link) K is called a 2-*bridge* knot (or link) if K has a diagram D that has only 2 local maximal (and 2 local minimal) points. See Fig. 5.1.

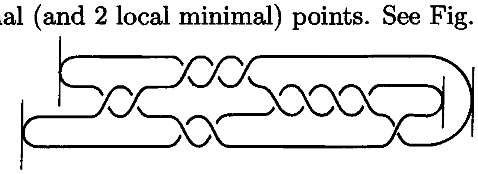


Figure 5.1

A 2-bridge knot (or link) K is characterized by two non-zero coprimes, α and β, where $-\alpha < \beta < \alpha$, or equivalently a rational number $-1 < \beta/\alpha < 1$. K is called a 2-bridge knot (or link) of *type* (α, β) and is denoted by $B(\alpha,\beta)$. K is a knot if and only if α is odd.

(3) **Diagram of $B(\alpha,\beta)$**

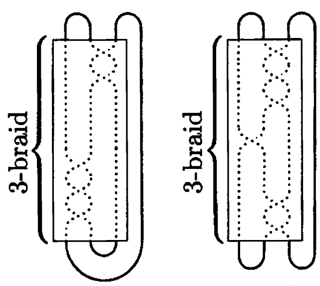
Given a rational number β/α, a 2-bridge knot (or link) $B(\alpha,\beta)$, $\gcd(\alpha,\beta) = 1$, is constructed in the following way. First express β/α as a continued fraction:

$$\frac{\beta}{\alpha} = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots - \cfrac{1}{a_{m-1} - \cfrac{1}{a_m}}}}}}$$

$$= [a_1, a_2, \cdots, a_m],$$

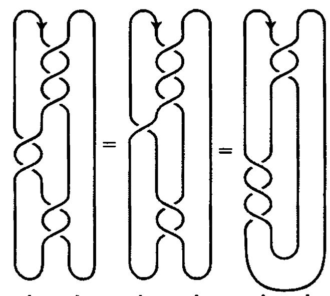
where $a_i \neq 0$. This expression is not unique. To $[a_1, a_2, \ldots, a_m]$ associate a 3-braid $\gamma = \sigma_2^{a_1}\sigma_1^{a_2}\sigma_2^{a_3}\sigma_1^{a_4}\ldots\sigma_1^{a_m}$ if m is even, or $\gamma = \sigma_2^{a_1}\sigma_1^{a_2}\sigma_2^{a_3}\sigma_1^{a_4}\ldots\sigma_2^{a_m}$, if m is odd.

Finally, join the top and bottom as is shown in Fig. 5.2.



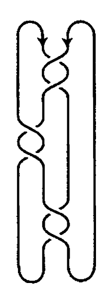
(a) $m \equiv 0 \pmod 2$ (b) $m \equiv 1 \pmod 2$

Figure 5.2



[3, 2, 2] [3, 1, −2] [2, −3]

Figure 5.3



[2, −2, −2]

Figure 5.4

The diagram of a knot (or link) thus obtained is a diagram of $B(\alpha, \beta)$. It is shown that no matter what continued fractions of β/α is used, we obtain the same (i.e. equivalent) knot (or link).

Example 5.1 (1) Since $\frac{3}{7} = [3, 2, 2] = [3, 1, -2] = [2, -3]$, three diagrams in Fig. 5.3 represent the same 2-bridge knot $B(7, 3)$.
(2) Since $\frac{3}{8} = [2, -2, -2]$, Fig. 5.4 depicts a 2-bridge link $B(8, 3)$.

(4) **Classification**

The orientation of $B(\alpha, \beta)$ is given as follows. The second string is always oriented downward (see Fig. 5.3). If $B(\alpha, \beta)$ is a link, then the third string forms one component and is oriented downward (see Fig. 5.4). Thus, we obtain an oriented 2-bridge knot (or link) $B(\alpha, \beta)$. These knots (or links) are classified by the following theorem.

Theorem 5.1 [Sc] *For oriented 2-bridge knots (or links), $B(\alpha, \beta)$ is equivalent to $B(\alpha', \beta')$ if and only if*
(i) $\alpha = \alpha'$ and $\beta \equiv \beta'$, (mod α), or
(ii) $\alpha = \alpha'$ and $\beta\beta' \equiv 1$ (mod α).

Example 5.2 (1) $B(3, 1) = B(3, -2)$ (2) $B(7, 4) = B(7, -3) = B(7, -5)$.

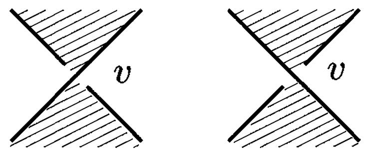
Theorem 5.2 *A 2-bridge knot $B(\alpha, \beta)$ is invertible, and $B(\alpha, \beta)$ is amphicheiral if and only if $\beta^2 \equiv -1$ (mod α).*

6. Minkowski units

In the following three sections, we discuss three classical knot invariants. As was stated in Introduction, the Minkowski unit is one of the earliest numerical invariants in knot theory. Since its value is either $+1$ or -1, it cannot be applied on the classification problem, but it gives an elegant solution to the amphicheirality problem.

(1) **Goeritz matrix**

Let D be a diagram of a knot (not a link) K. Then D divides $\mathbb{R}^2$ into finitely many domains, $D_1, D_2, \ldots, D_n$, one of which is unbounded, say D_n. We classify these domains into two classes, black and white in such a way that no domains of the same colour have edges in common. For convenience, we assume that D_n is a black domain. (This assumption is not a serious restriction, because any domain can be deformed into an unbounded domain by a suitable deformation of K.) Now we define an *index* ε of a crossing point v of D, where $\varepsilon(v) = \pm 1$ according to Fig. 6.1.



$\varepsilon(v) = 1 \qquad \varepsilon(v) = -1$

Figure 6.1

Here, a shaded area indicates a black domain and the orientation of K is irrelevant. Let $\{W_1, W_2, \dots, W_{m+1}\}$ be the set of all white domains. Using these white domains, we define an $(m+1) \times (m+1)$ integer symmetric matrix $A = (a_{i,j})$, $1 \le i, j \le m+1$.

For $i \neq j$, $-a_{i,j} = \sum \varepsilon(v)$, where the summation runs over all crossing points of D that are common to W_i and W_j. For $i = 1, 2, \dots, m+1, a_{i,i} = -\sum_{j=1, i\neq j}^{m+1} a_{i,j}$. This (integer symmetric) matrix A is called the *Goeritz matrix* of a knot K. Obviously, A is singular, so we eliminate one row, say the last row, and the last column from A to obtain an $m \times m$ matrix $\hat{A}$. Usually, $|\det \hat{A}|$ is called the *determinant* of a knot K. It is a knot invariant. However, the matrix $\hat{A}$ itself is not a knot invariant, since $\hat{A}$ depends on a diagram D. It is known that $|\det \hat{A}| \equiv 1 \pmod 2$.

(2) **Minkowski unit**

Take an odd prime p. If $p \nmid \det \hat{A}$, then we define the Minkowski unit $C_p(K)$ of K at p to be 1. So let $p | \det \hat{A}$. Suppose that there is an integer unimodular matrix B with which we can diagonalize $\hat{A}$ as follows, where $p \nmid b_j, j = 1, 2, \dots, m$:

$$B^T \hat{A} B = \begin{bmatrix} b_1 & & & & & & 0 \\ & b_2 & & & & & \\ & & \ddots & & & & \\ & & & b_k & & & \\ & & & & pb_{k+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & pb_m \end{bmatrix} \pmod{p^2},$$

where B^T denotes the transpose of B.

Then we define $C_p(K) = \left(\frac{(-1)^{[\frac{m-k}{2}]} b_{k+1} b_{k+2} \cdots b_m}{p}\right)$, where $\left(\frac{q}{p}\right)$ denotes Legendre symbol. The Minkowski unit $C_p(K)$ is an invariant of a knot K. For a link L, we can define the Goeritz matrix A in the same way, but $\det \hat{A}$ may be 0. Therefore, we need a different approach to define $C_p(L)$. See [Mu2].

Example 6.1 The Goeritz matrix of a knot K in Fig. 6.2 (a) is given by

$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, and hence $\det\hat{A} = 3$.

Therefore, $C_p(K) = 1$ for $p \neq 3$. Let $p = 3$. Then $\hat{A} \sim \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ (mod 9), and hence, $C_3(K) = \left(\frac{-1}{3}\right) = -1$. On the other hand, the Goeritz matrix of the mirror image K^* of K is $-A$. See Fig. 6.2 (b). Therefore, $C_3(K^*) = 1$. Since $C_3(K) \neq C_3(K^*)$, we conclude that K is not equivalent to K^*, and K is not amphicheiral.

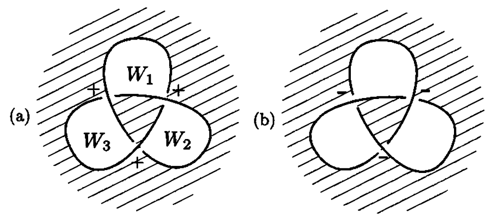


Figure 6.2

7. Seifert Matrix

In 1934, H.Seifert defined a new integer matrix N for a knot K. This matrix N is not necessarily symmetric and may be singular. However, using N, we can define two important invariants, called the *signature* and the *Alexander polynomial* of a knot K.

(1) **Seifert surface**

A connected orientable surface that bounds a given oriented knot (or link) K is called a *Seifert surface* of K. For an arbitrary knot (or link), such a surface always exists. In fact, Seifert gave an algorithm by which one can construct such a surface from a diagram D of K. Given a diagram D of K, first draw a small circle with one of the crossing point of D as its center. This circle intersects D at four points, say a, b, c and d. See Fig. 7.1 (a).

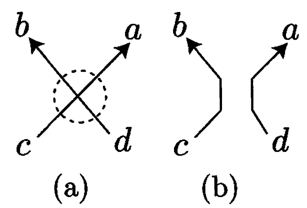


Figure 7.1

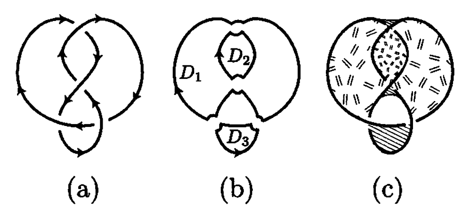


Figure 7.2

Then as is shown in Fig. 7.1 (b), splice this crossing point and connect a and d, and b and c. In this way we can remove the crossing point of D that lies within the circle. Now apply this process at every crossing point of D, and we remove all the crossing points from D. Then D becomes decomposed into several simple closed curves, and we span each simple closed curve by a disk. For the knot in Fig. 7.2 (a), there are three disks D_1, D_2, D_3. See Fig. 7.2 (b).

Finally, we attach a half-twisted band at the place of D that corresponds to a crossing point before they are removed. See Fig. 7.2 (c). Thus we obtain a connected, orientable surface S that bounds K. A shaded areas denotes the *face (or front)* of the surface S and a dotted area the *back* of S. (If K is a link, we alter

K, if necessary, in such a way that the projection of K is connected, then by the above method we can also obtain a connected orientable surface.)

By this simple process, we can guarantee the existence of at least one Seifert surface for any knot (or link). However, the surface obtained by this algorithm (called *Seifert algorithm*) may be unnecessarily complicated. For a Seifert surface S, we denote by $g(S)$ the genus of S. The genus of S is evaluated as $\frac{1}{2}(2-n-\chi(S))$, where n is the number of component of K and $\chi(S)$ denotes the Euler Characteristic of S. The minimum genus of all Seifert surface for a given knot (or link) K is an invariant of K. For an arbitrary knot (or link), there does exist an algorithm to actually calculate its genus, but it is exceedingly difficult to implement. In truth, to calculate its genus for an arbitrary knot (or link) is a difficult undertaking.

Remark 7.1 If a torus T is a boundary of a tubular neighbourhood of a knot K, then a longitude of T can be obtained as the intersections of a Seifert surface of K and T.

(2) **Linking number**

The *linking number*, denoted by $\text{lk}(\lambda, \mu)$, between two disjoint oriented simple closed curves λ and μ in $\mathbb{R}^3$ is the simplest, but important link invariant of a 2-component link $L = \lambda \cup \mu$. The linking number $\text{lk}(\lambda, \mu)$ is evaluated as follows. Consider a diagram D of a link L. See Fig. 7.3.

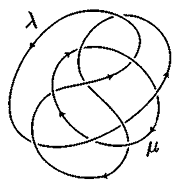


Figure 7.3

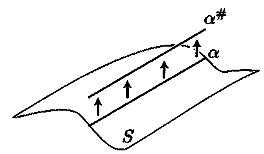


Figure 7.4

Let n_+ (or n_-) denote the number of times μ passes *under* λ from the right to left, (or the left to right). Then $\text{lk}(\lambda, \mu) = n_+ - n_-$. For the link L in Fig. 7.3, $\text{lk}(\lambda, \mu) = 1$. It is known that $\text{lk}(\lambda, \mu) = \text{lk}(\mu, \lambda)$.

Now we return to a Seifert surface.

(3) **Seifert Matrix**

Let S be a Seifert surface of a knot (or link) K and g the genus of S. Then the first homology group $H_1(S;\mathbb{Z})$ of S is a free abelian group of rank $2g+n-1$, where n is the number of components of K. For convenience, denote $m = 2g+n-1$. Now we can choose m oriented simple closed curves $\alpha_1, \alpha_2, \dots, \alpha_m$ on the face of S in such a way that (a) For $i \neq j, \alpha_i$ and α_j intersect only at a finite number of points. (b) Homology classes $[\alpha_1], \dots, [\alpha_m]$ form a basis for $H_1(S;\mathbb{Z})$. Using these curves, $\alpha_1, \dots, \alpha_m$, we define an $m \times m$ integer matrix $N = (c_{i,j}), 1 \leq i, j \leq m$, where $c_{i,j} = \text{lk}(\alpha_i^\#, \alpha_j)$, Here $\alpha_i^\#$ is a simple closed curve in $\mathbb{R}^3$ that is obtained from α_i by lifting a bit in the positive normal direction, see Fig. 7.4. The orientation of $\alpha_i^\#$ is induced from that of α_i. Even if α_i and α_j may intersect, $\alpha_i^\#$ and α_j never meet, since $\alpha_i^\#$ is no longer on S. The matrix N is called the *Seifert matrix* of K. The matrix N is an integer matrix, but it may be singular, and in general it is not symmetric. Further, N itself is not an invariant of K, since N depends on a Seifert surface.

Example 7.1 Consider a Seifert surface S depicted in Fig. 7.5 (a). Choose α_1 and α_2 as is shown in Fig. 7.5 (a). Then the Seifert matrix $N = (c_{i,j})$ is given as follows. See Fig. 7.5 (b)-(e). $c_{1,1} = \mathrm{lk}(\alpha_1^{\#}, \alpha_1) = -1$, $c_{1,2} = \mathrm{lk}(\alpha_1^{\#}, \alpha_2) = 0$, $c_{2,1} = \mathrm{lk}(\alpha_2^{\#}, \alpha_1) = 1$, $c_{2,2} = \mathrm{lk}(\alpha_2^{\#}, \alpha_2) = 0$, and hence, $N = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

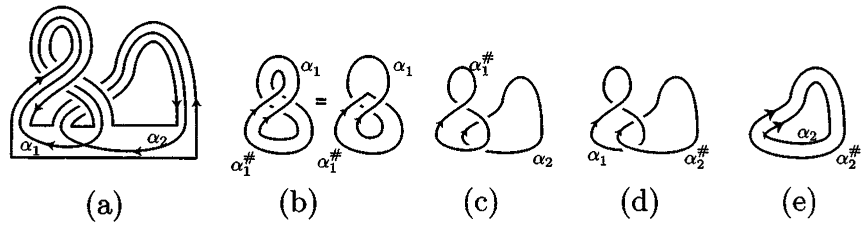


Figure 7.5

(4) **Signature**

Let N be an $m \times m$ Seifert matrix of an (oriented) knot (or link) K. Then, $N + N^T$ is a symmetric integer matrix. If K is a knot, it is not singular and $|\det(N + N^T)| = |\det \hat{A}|$, where$\hat{A}$ was defined in Section 6. Furthermore, it is shown [T2, Mu3] that the signature of $N + N^T$ is an invariant of a (oriented) knot (or link) K that is called the *signature* of K, and denoted by $\sigma(K)$. Here, the signature of $N + N^T$ is defined as follows: First, diagonalize $N + N^T$ over $\mathbb{Q}$ by means of some unimodular matrix B. Let $b_1, b_2, \ldots, b_m$ be diagonal elements. Then signature of $N + N^T$ is defined as the number of positive b_i's minus the number of negative b_j's.

Theorem 7.1 *Let K be an oriented knot and K^* the mirror image of K. Then $\sigma(K^*) = -\sigma(K)$.*

Therefore, if K is amphicheiral, then $\sigma(K) = 0$. The converse is not true. Let $\hat{K}$ be the knot obtained from K by reversing the orientation. Then, $\sigma(\hat{K}) = \sigma(K)$.

(5) **Alexander polynomial**

The Alexander polynomial $\Delta_K(t)$ of a knot K is redefined as, up to $\pm t^k$, $\Delta_K(t) = \det(N - tN^T)$. The original definition by Alexander was combinatorial [A2]. The Alexander polynomial of a knot is characterized in the following theorem due to Seifert.

Theorem 7.2 [Se] *If an integer polynomial $f(t)$ satisfies the following three conditions: (a) the degree of $f(t)$ is even, $2r$ say, (b) $f(t)$ is symmetric, i.e. $f(t) = t^r f(t^{-1})$, (c) $f(1) = 1$, then $f(t)$ is the Alexander polynomial of some knot in $\mathbb{R}^3$. The converse is also true.*

For a link L, the polynomial $\Delta_L(t) = \det(N - tN^T)$ is also an invariant of L, up to $\pm t^k$, where N is a Seifert matrix of L. This is called the *reduced* Alexander polynomial of L. From these definitions, we see

Theorem 7.3 *The degree of the (reduced) Alexander polynomial of a knot (or link) K is at most $m = 2g + n - 1$.*

In the next section, we will define the Alexander polynomial of a link, and also will discuss other formulations of the Alexander polynomial.

8. Alexander polynomials

Since Fox's free derivatives [F1] will be the most effective tool to define the Alexander polynomial for knots and links simultaneously, first we explain this operation briefly.

(1) **Free derivatives**

Let F be a free group generated freely by $x_1, x_2, \ldots, x_n$, and let $\mathbb{Z}F$ denote the integer group ring. With a word $w = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \ldots x_{j_k}^{\epsilon_k}$ in F, we associate the *free derivatives* $\partial w/\partial x_1, \partial w/\partial x_2, \ldots, \partial w/\partial x_n$ defined as follows: $\partial w/\partial x_j = \epsilon_1 \delta_{j,j_1} x_{j_1}^{(\epsilon_1-1)/2} + \epsilon_2 \delta_{j,j_2} x_{j_1}^{\epsilon_1} x_{j_2}^{(\epsilon_2-1)/2} + \ldots + \epsilon_k \delta_{j,j_k} x_{j_1}^{\epsilon_1} \ldots x_{j(k-1)}^{\epsilon_{j(k-1)}} x_{j_k}^{(\epsilon_{j_k}-1)/2}$.

For example, if $w = x_1 x_2^{-1} x_1 x_1 x_2 x_1^{-1} x_2, \partial w/\partial x_1 = 1 + x_1 x_2^{-1} + x_1 x_2^{-1} x_1 - x_1 x_2^{-1} x_1 x_1 x_2 x_1^{-1}, \partial w/\partial x_2 = -x_1 x_2^{-1} + x_1 x_2^{-1} x_1 x_1 + x_1 x_2^{-1} x_1 x_1 x_2 x_1^{-1}$, where $1 = 1e$, e being the identity of F. Therefore, the free derivatives are mappings from F to $\mathbb{Z}F$, but they are obviously extended to mappings from $\mathbb{Z}F$ to $\mathbb{Z}F$. The following formulas are easily verified.

(a) $\partial 1/\partial x_j = 0, \partial x_j/\partial x_k = \delta_{j,k}$, and $\partial x_j^{-1}/\partial x_j = -x_j^{-1}$, for any $j = 1, 2, \ldots, n$,

(b) $\partial(uv)/\partial x_j = \partial u/\partial x_j + u\partial v/\partial x_j$, for any words u and v in F and $j = 1, 2, \ldots, n$,

(c) (Fundamental formula) $w - 1 = \displaystyle\sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1)$, for any word w in F.

Note that since F is not commutative, $\dfrac{\partial w}{\partial x_j}(x_j - 1) \neq (x_j - 1)\dfrac{\partial w}{\partial x_j}$.

(2) **Alexander polynomials**

Let K be a link of r components, $r \geq 1$. Consider a Wirtinger presentation of the group $G(K)$ of K. $G(K) = \langle x_{i,j} | R_{i,j} = 1 \rangle, 1 \leq i \leq r, 1 \leq j \leq m_i$. See Section 2 (5). Compute $\partial R_{i,j}/\partial x_{k,\ell}$ for all i, j and k, ℓ, and obtain a square matrix $M = [\partial R_{i,j}/\partial x_{k,\ell}]$ of order $\displaystyle\sum_{i=1}^{r} m_i$, called the *Jacobian*. Now let F be the free group of rank $\displaystyle\sum_{i=1}^{r} m_i$ freely generated by $\{x_{i,j}\}, 1 \leq i \leq r, 1 \leq j \leq m_i$, and $\phi : F \to G$ be a natural homomorphism so that Kerϕ is the normal closure of $\{R_{i,j}\}$. Further, let $\psi : G \to G/G' = H$ be the abelianization of G, and hence, H is a free abelian group of rank r generated by $\{x_{i,1}\}, 1 \leq i \leq r$. For convenience, we use t_i for $x_{i,1}$, but if $r = 1$, we use t for x_1. Now evaluate M at $\phi\psi$, and hence we obtain a square matrix $M^{\phi\psi}$ of order $\sum m_i$, where an entry of $M^{\phi\psi}$ is an integer Laurent polynomial in $t_1, t_2, \ldots, t_r$. Note that $M^{\phi\psi}$ is singular.

Definition 8.1 The g.c.d. of all minors of $M^{\phi\psi}$ of order $\sum m_i - 1$ is called the *Alexander polynomial* of K, denoted by $\Delta(t_1, t_2, \ldots, t_r)$.

This is an invariant of K up to a unit of $\mathbb{Z}H$, i.e. $\pm t_1^{k_1} \ldots t_r^{k_r}$. If $r = 1$, this polynomial coincides with $\det(N - tN^T)$, where N is a Seifert matrix of K studied in section 7.

The Alexander polynomial may possibly be 0 for a link, but for a knot, it is never 0, since $|\Delta(1)| = 1$. Further, if $r > 1$, set $\widehat{\Delta}_K(t) = \Delta_K(t, t, \ldots, t)$ that is obtained from the Alexander polynomial of K by substituting t for every $t_i, 1 \leq i \leq r$. It is known that$(1 - t)\widehat{\Delta}_K(t) = \det(N - tN^T)$. Therefore $(1 - t)\widehat{\Delta}_K(t)$ is the reduced Alexander polynomial of a link K.

Remark 8.1 If $G(K)$ has a presentation such that the number of relations is less than $\sum m_i - 1$, then $\Delta_K(t_1, t_2, \ldots, t_r)$ is defined to be 0.

Example 2.2 (continued) From the presentation of $G(K)$, we obtain
$\partial R_1/\partial x_1 = 1, \partial R_1/\partial x_2 = -x_1 x_3 x_2^{-1}, \partial R_1/\partial x_3 = x_1 - x_1 x_3 x_2^{-1} x_3^{-1}$,
$\partial R_2/\partial x_1 = x_2 - x_2 x_1 x_3^{-1} x_1^{-1}, \partial R_2/\partial x_2 = 1, \partial R_2/\partial x_3 = -x_2 x_1 x_3^{-1}$.

Therefore, we have, $M^{\phi\psi} = \begin{bmatrix} 1 & -t & t-1 \\ t-1 & 1 & -t \\ * & * & * \end{bmatrix}$, and $\Delta_K(t) = 1 - t + t^2$.

(We do not need to evaluate $\partial R_3/\partial x_i$.)

Example 2.3 (continued) By Remark 8.1, $\Delta_L(t_1, \ldots, t_r) = 0$.

Example 2.4 (continued) Since $R_{1,1} = x_{1,1}x_{2,1}x_{1,1}^{-1}x_{2,1}^{-1}$, we have $\partial R_{1,1}/\partial x_{1,1} = 1 - x_{1,1}x_{2,1}x_{1,1}^{-1}$ and $\partial R_{1,1}/\partial x_{2,1} = x_{1,1} - x_{1,1}x_{2,1}x_{1,1}^{-1}x_{2,1}^{-1}$, and hence, $M^{\phi\psi} = \begin{bmatrix} 1-t_2 & t_1-1 \\ * & * \end{bmatrix}$. Therefore, $\Delta_L(t_1, t_2) = 1$.

In the rest of this section, we discuss the Alexander polynomials of torus knots and 2-bridge knots.

(3) **Torus knots**

With a Seifert matrix N of a torus knot $T(p,q)$, the calculation of $\det(N - tN^T)$ is cumbersome, but the final form is neat.

Theorem 8.1 *(1) If $pq = 0$, then $\Delta_{T(p,q)}(t) = 1$.*
(2) If $pq \neq 0$, then $\Delta_{T(p,q)}(t) = (t^{pq}-1)(t-1)/(t^p-1)(t^q-1)$.

Using Theorem 8.1, we can prove Theorem 4.4 (ii).

(4) **2-bridge knots**

The calculation of the Alexander polynomial of a 2-bridge knot $B(\alpha, \beta)$, using a Seifert matrix, is also quite messy. However, a different approach makes it possible to write down the Alexander polynomial of $B(\alpha, \beta)$ in one form. See Theorem 9.5 in Section 9. Finally, we state a few properties of the Alexander polynomial of $B(\alpha, \beta)$. Since $\Delta_{B(\alpha,\beta)}(t) = \Delta_{B(\alpha,\beta)}(t) = \Delta_{B(\alpha,\alpha-\beta)}(t)$ and α is odd, we may assume that β is even. Consider the even continued fraction of a rational β/α:

$$\frac{\beta}{\alpha} = \cfrac{1}{2a_1 - \cfrac{1}{2a_2 - \cfrac{1}{\ddots - \cfrac{1}{2a_{m-1} - \cfrac{1}{2a_m}}}}}$$
$$= [2a_1, 2a_2, \cdots, 2a_m],$$

where $a_k \neq 0, 1 \leq k \leq m$. Then we have:

Theorem 8.2 [Kan] *(1) the degree of $\Delta_{B(\alpha,\beta)}(t)$ is exactly m,*
(2) $|\Delta_{B(\alpha,\beta)}(0)| = |a_1 a_2 \ldots a_m|$. Therefore, $\Delta_{B(\alpha,\beta)}(t)$ is monic if and only if $|a_1| = |a_2| = \cdots = |a_m| = 1$, namely, $\beta/\alpha = [\pm 2, \pm 2, \ldots, \pm 2]$.

9. Signature

In this section, we evaluate the signatures for torus knots and 2-bridge knots.

(1) **Torus knots**

The calculation of the signatures for torus knots is not easy. The earliest attempt was made by Hirzebruch in 1968, and he proved:

Theorem 9.1 [HizM] *Let $T(p,q)$ be a torus knot of type (p,q), where $p > 0$ and $q > 0$. Suppose that p and q both are odd. Then*

$$\sigma(T(p,q)) = 2\{\frac{(p-1)}{2}\frac{(q-1)}{2} + N_{p,q} + N_{q,p}\},$$

where $N_{p,q} = |\{x | 1 \leq x \leq (q-1)/2, ((px/q)) > 0\}|$, and

$$((y)) = \begin{cases} y - [y] - 1/2, & \text{if } y \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{if } y \in \mathbb{Z}. \end{cases}$$

Example 9.1 $\sigma(T(5,3)) = 2\{\frac{4}{2}\ \frac{2}{2} + N_{5,3} + N_{3,5}\} = 8$, since$N_{5,3} = N_{3,5} = 1$.

Hirzebruch also proves the following reduction formula.

Theorem 9.2 *Under the same assumptions of Theorem 9.1, the following formula holds:* $\sigma(T(p+2q,q)) = \sigma(T(p,q)) - (q^2-1)$.

In 1981, we proved the following complete reduction formula for $\sigma(T(p,q))$.

Theorem 9.3 [GLM] *Write* $\sigma(p,q)$ *for* $\sigma(T(p,q))$. *Assume* $p > 0$ *and* $q > 0$.

(1) $\sigma(p,q) = \sigma(q,p)$. *Therefore, we may assume further that* $0 < q < p$.

(2) Case $2q < p$.

(i) If $q \equiv 1 \pmod 2$, *then* $\sigma(p,q) = \sigma(p-2q,q) + q^2 - 1$.

(ii) If $q \equiv 0 \pmod 2$, *then* $\sigma(p,q) = \sigma(p-2q,q) + q^2$.

(3) Case $q < p < 2q$.

(i) If $q \equiv 1 \pmod 2$, *then* $\sigma(p,q) + \sigma(2q-p,q) = q^2 - 1$.

(ii) If $q \equiv 0 \pmod 2$, *then* $\sigma(p,q) + \sigma(2q-p,q) = q^2 - 2$.

(4) For $p > 0$, $\sigma(p,2) = p - 1$ *and* $\sigma(p,1) = 0$.

Example 9.2 For $p = 4$ and $q = 3$, we have from (3)(i) $\sigma(4,3) + \sigma(2,3) = 8$ and since $\sigma(2,3) = \sigma(3,2) = 2$, we see $\sigma(4,3) = 6$.

(2) **2-bridge knots**

In contrast to torus knots, an evaluation of the signature of a 2-bridge knot is straightforward. Let $B(\alpha,\beta)$ be a 2-bridge knot, where α is odd. Since $B(\alpha,-\beta)$ is the mirror image of $B(\alpha,\beta)$, it follows that $\sigma(B(\alpha,-\beta)) = -\sigma(B(\alpha,\beta))$, and hence, we may assume that $\alpha > \beta > 0$. Further, we may assume that β is odd. In fact, Theorem 5.1 yields $\sigma(B(\alpha,\beta)) = \sigma(B(\alpha,\beta-\alpha)) = -\sigma(B(\alpha,\alpha-\beta))$. Now consider the sequence $E = \{\beta, 2\beta, 3\beta, \dots, (\alpha-1)\beta\}$. Take representatives $\overline{k\beta}$ of $k\beta \pmod{2\alpha}$ in $-\alpha < \overline{k\beta} < \alpha$, and obtain a new sequence $\bar{E} = \{\overline{\beta}, \overline{2\beta}, \overline{3\beta}, \dots, \overline{(\alpha-1)\beta}\}$. Let ϵ_k be the sign of $\overline{k\beta}$, i.e. $\epsilon_k = \overline{k\beta}/|\overline{k\beta}|$.

Theorem 9.4 [Sh] *If* α *and* β *are odd and* $\alpha > \beta > 0$, *then* $\sigma(B(\alpha,\beta)) = \sum_{k=1}^{\alpha-1} \epsilon_k$.

Example 9.3 For $B(5,3)$, we have $E = \{3,6,9,12\}$ and $\bar{E} = \{3,-4,-1,2\}$ and therefore, $\sigma(B(5,3)) = 0$.

Finally, the Alexander polynomial of a 2-bridge knot $B(\alpha,\beta)$ can be calculated in the following simple form:

Theorem 9.5 [HiM1] *If* α *and* β *are both odd, and* $\alpha > \beta > 0$, *then:*

$$\Delta_{B(\alpha,\beta)}(t) = \sum_{k=0}^{\alpha-1} (-1)^k t^{q_k}, \textit{where } q_k = \sum_{i=0}^{k} \epsilon_i, \textit{and } \epsilon_0 = 1$$

10. Fibred knots

(1) **Commutator subgroup and Augmentation subgroup**

Almost the same time when Alexander defined his polynomial, Reidemeister also introduced the same polynomial from a different point of view.

Let K be a knot (not a link) and $G = G(K)$ the group of K. Let G' be the commutator subgroup of G. Then the infinite cyclic group $G/G' \cong Z = \langle t|-\rangle$ acts on G' by conjugation, and G'/G'' becomes a finitely generated $\mathbb{Z}[t^{\pm}]$-module with the relation matrix Λ. Then Reidemeister showed that $\det\Lambda$ is an invariant of K, up to $\pm t^k$, and in fact it is the Alexander polynomial of a knot K. For an (oriented) r-component link $L, r > 1$, the commutator subgroup G should be replaced by the kernel of $\phi\psi\tau$, where $\tau : H \to \langle t|-\rangle$, and $\tau(t_i) = t, 1 \le i \le r$. The subgroup $A = \ker \phi\psi\tau$ is called the *augmentation subgroup* of the group of L. Then the same argument above shows that the reduced Alexander polynomial $(1-t)\hat{\Delta}(t)$ is given by $\det \Lambda$. It is important to note that the augmentation subgroup A depends on the orientation of each component. If one of the components reverses its orientation, we have a different augmentation subgroup and hence the different (reduced) Alexander polynomial.

(2) **Fibred knots (or links)**

This reformulation suggests that the Alexander polynomial may provide us some information on the structure of G/G''. In fact, we have

Theorem 10.1 [Ra] *Let $\Delta_K(t)$ be the Alexander polynomial of a knot K. Then $\Delta_K(t)$ is monic if and only if G'/G'' is finitely generated, and then G'/G'' is a free abelian group of rank d, where d is the degree of $\Delta_K(t)$.*

On the other hand, L. Neuwirth studied the structure of the commutator subgroup G' of the group of a knot K, and, in particular, he proved

Theorem 10.2 [N] *If the commutator subgroup G' of the group of a knot K is finitely generated, then G' is a free group of rank d, the degree of the Alexander polynomial.*

Later, the property that G' be finitely generated was characterized topologically by J.Stallings.

Theorem 10.3 [St] *Let K be a knot. Suppose that the commutator subgroup of $G(K)$ is finitely generated. Then the complement of a small tubular neighbourhood of K in S^3 is the total space of a fibre space with base space a circle and with fibre a Seifert surface of K. The converse is also true. We call such a knot a fibred knot.*

Remark 10.1 For a link in S^3, analogous theorems to Theorems 10.2 and 10.3 hold if the augmentation subgroup A replaces the commutator subgroup. As an immediate consequence of Theorem 10.3, we obtain:

Theorem 10.4 *If K is a fibred knot, then the Alexander polynomial of K is monic. The converse is not true, however.*

The property that K be a fibred knot is a topological property, and hence, it is difficult to characterize this property by *algebraic* invariants. However, for 2-bridge knots or more generally for *alternating* knots, their Alexander polynomials characterize fibred knots. See Theorem 10.5 below. An alternating knot is a knot that has at least one *alternating* diagram. A diagram is called *alternating* if over-passing and under-passing alternate while moving along a diagram. Fig. 1.1 is an alternating diagram, while Fig. 2.1 is not an alternating diagram, but the knot itself is an alternating knot, since it has an alternating diagram. A 2-bridge knot is an alternating knot. Alternating knots form a large and special class in knot theory, and the classification problem is completely solved for these knots [MeT]. But, not all knots are alternating. A torus knot $T(p,q)$ is alternating if and only if $\min\{|p|,|q|\} \le 2$. For alternating knots, the converse of Theorem 10.4 holds. Namely, we have:

Theorem 10.5 [Mu1] *Let K be an alternating knot. Then K is a fibred knot if and only if $\Delta_K(t)$ is monic.*

Recently, Gabai introduced some geometric methods by which one can decide whether many knots (or links) are fibred or not [G].

Remark 10.2 A trivial knot is a fibred knot, and all torus knots are fibred knots. A Hopf link is a fibred link.

11. Representation - Covering space

Let J_n denote a set of n elements, $\{1, 2, \ldots, n\}$, say, and $S(J_n)$ the group of all permutations of the set J_n.

(1) **Permutation Representation**

Let $G(K)$ be the group of a knot (or link) K in S^3. Let Φ be a transitive representation of $G(K)$ into $S(J_n)$ for some n. Corresponding to Φ is the covering space M of S^3 branched along K with the covering projection p (that we call simply the *branched covering space of* K in S^3). Two permutation representations Φ and Φ' are called *equivalent* if they differ by renumbering of J_n. Two covering spaces corresponding to Φ and Φ' are homeomorphic. Now if H is a subgroup of $G(K)$ with finite index n, then we obtain a transitive representation Φ of $G(K)$ into $S(J_n)$. (Any two such representations are equivalent.) This representation will be called a representation corresponding to the subgroup H. In fact, H is obtained as the stabilizer of some element in J_n under Φ.

(2) **Cyclic covering space - Homology invariants**

By taking various representations of $G(K)$, we obtain various compact 3-manifolds M as branched covering spaces of K. Since these manifolds are invariants of K, we can deduce many knot invariants from these manifolds. For example, the first homology group $H_1(M; \mathbb{Z})$ is one of many classical invariants that were already studied in 1930's. For example, the torsion number and Betti number of $H_1(M; \mathbb{Z})$ provide us handy and neat numerical invariants. In particular, the branched covering space of K associated with the cyclic representation $\Phi : G(K) \to \{(12 \cdots n)\} \subset S(J_n)$ was the subject of investigation from the early stage of knot theory. In fact, we have

Theorem 11.1 *Let M_n be the n-fold cyclic branched covering space of a knot K. Let $\beta_1(K)$ and $\tau_1(K)$ denote respectively the first Betti number and torsion number of $H_1(M; \mathbb{Z})$. Then, if $\beta_1(K) = 0$, then $\tau_1(K) = \prod_{i=1}^{n-1} \Delta_K(\xi^i)$, where ξ is a primitive nth root of unity.*

Note that if $n = 2$, then $\beta_1(K) = 0$ for a knot K.

For a knot, any abelian covering is necessarily a cyclic covering. However, for a link L, there are infinitely many non-cyclic abelian coverings. But, if $\beta_1(L) = 0$, there is a formula that evaluates the torsion number of $H_1(M; \mathbb{Z})$ in terms of the Alexander polynomial of L and those of the various sublinks of L [MaM]. These formulas give us the torsion number of $H_1(M; \mathbb{Z})$, but do not give any information on the structure of $H_1(M; \mathbb{Z})$. The structure of $H_1(M; \mathbb{Z})$ is generally complex. A few classical results are known for the case of cyclic coverings of knots [Pl]. Later, we will discuss some results for the case of links.

(3) **Dihedral representations**

Besides abelian representations, representations of the knot group on the dihedral groups D_p of order $2p, p$ odd, have been studied extensively. The following theorem will easily be proved.

Theorem 11.2 *Let $D_p = \langle a, b | a^2 = b^2 = (ab)^p = 1\rangle$ be the dihedral group of order $2p$, where p is odd. Then the knot group $G(K)$ has an (irregular) representation $\Phi_p : G(K) \to D_p \subset S(J_p)$ in such a way that Φ_p maps every meridian element of K to an element of order 2 in D_p if and only if $\Delta_K(-1) \equiv 0 \pmod p$.*

It is shown that if K is a 2-bridge knot, then the branched covering space corresponding to Φ_p is simply connected [F2], but in fact it is S^3 [Bu2]. One of the reasons to study (irregular) branched covering spaces of knots (or links) is hopefully to obtain a simply connected closed 3-manifold other than S^3. But, up to the present, nobody has found such a 3-manifold. Besides the dihedral coverings, various coverings have been studied, see [Ri], [HaM2].

(4) **Covering linkage invariants**

As was seen in the above subsection, the covering spaces themselves may not be significant invariants of a knot. However, even if the covering space M is simple, like S^3, we can deduce from M some important invariants of K. Let $\Phi : G(K) \to S(J_n)$ be a transitive representation of $G(K)$. Suppose that the branched covering space M corresponding to Φ is a homology 3-sphere, i.e. $H_1(M;\mathbb{Q}) = 0$ and, further, $p^{-1}(K)$, a lift of K in M is an r-component link $\tilde{K} = \tilde{K}_1 \cup \tilde{K}_2 \cup \cdots \cup \tilde{K}_r, r \geq 2$. Then the linking numbers $\mathrm{lk}(\tilde{K}_i, \tilde{K}_j) \pmod q, 1 \leq i \neq j \leq r$, are well-defined and they are invariants of K, where q is the order of $H_1(M;\mathbb{Z})$. They are called the *covering linkage invariants* associated to Φ. These invariants are sometimes very powerful tools to distinguish between two knots. In fact, two 2-bridge knots that have the same Alexander polynomials are distinguished by the covering linkage invariants associated to (irregular) dihedral representations Φ_p [BS]. The following theorem is one of many results about the covering linkage invariants.

Theorem 11.3 [Pe] *Suppose that the group of a 2-bridge knot K has an (irregular) representation on $D_p \subset S(J_p)$, where p is odd. Then the lift of K in the irregular dihedral covering space M of K is a link $\tilde{K}$ of $(p+1)/2$ components $\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_{(p+1)/2}$, and for $i \neq j, \mathrm{lk}(\tilde{K}_i, \tilde{K}_j) \equiv 2 \pmod 4$.*

Evaluation of these covering linkage invariants for a knot or link associated to an arbitrary finite representation Φ is completely formulated [Ha] and various applications can be found in [HaM1], [HaM2], [Mu5], [Lu].

12. Modular representations of 2-bridge knot groups

In this section, we will discuss an unimodular representation of the group of a 2-bridge knot $B(\alpha, \beta)$. The information we obtained in this section will be used in the next section.

(1) **Unimodular representations**

Suppose there is an unimodular representation $\rho : G(K) \to SL(2,\mathbb{Z})$ such that $\rho(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, where m is one Wirtinger generator.

Such a representation ρ will be called a *canonical* representation. Since $PSL(2,\mathbb{Z}) \cong Z/(2) * Z/(3), \rho$ can be lifted to a homomorphism $\bar{\rho} : G(K) \to H = \langle a, b | a^2 = b^3\rangle$ [HaM2]. Now H is the group of a torus knot of type $(3,2)$ and $\Delta_{T(3,2)}(t) = 1 - t + t^2$, and hence, we have

Theorem 12.1 *If the group $G(K)$ of a knot K has a canonical unimodular representation $\rho : G(K) \to SL(2,\mathbb{Z})$, then $\Delta_K(t) \equiv 0 \pmod{1 - t + t^2}$.*

The converse of Theorem 12.1 is not true. For example, the Alexander polynomial of the 2-bridge knot $B(27,17)$ is $(1-t+t^2)(2-5t+2t^2)$, but its group does not have a canonical representation on $SL(2,\mathbb{Z})$. See Theorem 12.2.

(2) **Admissible form**

In this subsection, we characterize a 2-bridge knot $K = B(\alpha,\beta)$ that has a canonical representation on $SL(2,\mathbb{Z})$. First, we consider a continued fraction of β/α. Since the groups of 2-bridge knots $B(\alpha,\beta)$ and $B(\alpha,-\beta)$ are isomorphic, we may assume that $\alpha \equiv \beta \equiv 1 \pmod 2$ and $0 < \beta < \alpha$. Now express β/α as the continued fraction of the form:

$$\frac{\beta}{\alpha} = \cfrac{1}{2a_1 - \cfrac{1}{2a_2 - \cfrac{1}{\ddots - \cfrac{1}{2a_m - \cfrac{1}{c}}}}}$$

$$= [2a_1, 2a_2, \cdots, 2a_m, c],$$

where $a_i \neq 0$, c is odd and $|c| > 1$. This expression is called an *almost even continued fraction* of β/α, and this expression is unique. We say that β/α or $[2a_1, 2a_2, \ldots, 2a_m, c]$ is *admissible* if $G(B(\alpha,\beta))$ has a canonical representation on $SL(2,\mathbb{Z})$. Note that if $[2a_1, 2a_2, \ldots, 2a_m, c]$ is admissible, then $[-2a_1, -2a_2, \ldots, -2a_m, -c]$ is also admissible. The admissibility of an almost even continued fraction is completely determined by the following theorem.

Theorem 12.2 [HiM2] *Let $X = [2a_1, 2a_2, \ldots, 2a_m, c]$, c odd and $|c| > 1$, be the almost even continued fraction of β/α, where $\alpha > \beta > 0$ and $\alpha \equiv \beta \equiv 1 \pmod 2$. We call $m+1$ the length of X. (1) If X is admissible, then the length of X must be odd, i.e. m is even.*

(2) Suppose the length of X is 1. Then $[c]$ is admissible if and only if $c \equiv 3 \pmod 6$,

(3) Suppose the length of X is 3, i.e. $X = [2a_1, 2a_2, c]$. Then,

(i) $[6k, 2a_2, c], k \neq 0$, is admissible if and only if $[c]$ is admissible, i.e. $c \equiv 3 \pmod 6$.

(ii) $[6k+2, 2a_2, c], k \neq 0$, is admissible if and only if $2a_2 = -2$, and $[c-2]$ is admissible, i.e. $c \equiv 5 \pmod 6$.

(iii) $[6k+4, 2a_2, c], k \neq 0$, is admissible if and only if $2a_2 = 2$, and $[c-4]$ is admissible, i.e. $c \equiv 1 \pmod 6$.

(4) Suppose the length of $X \geq 5$.

(i) $[6k, 2a_2, \ldots, 2a_m, c], k \neq 0$, is admissible if and only if $[2a_3, 2a_4, \ldots, 2a_m, c]$ is admissible.

(ii) $[6k+2, 2a_2, \ldots, 2a_m, c], k \neq 0$, is admissible if and only if $2a_2 = -2$, and $[2a_3 - 2, 2a_4, \ldots, 2a_m, c]$ is admissible.

(iii) $[6k+4, 2a_2, \ldots, 2a_m, c], k \neq 0$, is admissible if and only if $2a_2 = 2$, and $[2a_3 - 4, 2a_4, \ldots, 2a_m, c]$ is admissible.

A proof needs a careful study of the modular diagram and a theorem due to Sakuma [Sa]. See also [GR] for a similar theorem to Theorem 12.2.

Example 12.1 Let K be a 2-bridge knot $B(27,17)$. Since $17/27 = [2,2,-2,3]$, it is not admissible, but $1-t+t^2$ divides $\Delta_K(t)$.

Example 12.2 Let $K = B(27,11)$. Since $11/27 = [2,-2,5]$, it is admissible.

13. Twisted Alexander polynomial

The twisted Alexander polynomial of a knot (or link) was defined by Wada [Wad] and Lin [Li] in early 1990's. This is a generalization of the Alexander polynomial, but it is a subtler invariant than the Alexander polynomial. In fact, it may distinguish between a knot and its mirror image. This is in general not possible by the Alexander polynomial.

(1) **Definition**

Originally, Wada defined this polynomial for a finitely presented group G with G/G' free. However, for simplicity, we consider only the group of a knot K. Let $\rho : G = G(K) \to GL(n, R)$ be a linear representation of the group of a knot K, where R is an unique factorization domain. Let $\alpha : G \to G/G' \cong \mathbb{Z} = \langle t | - \rangle$ be an abelianization. Then ρ and α can be extended to $\tilde{\rho} : \mathbb{Z}G \to M_n(R)$, and $\tilde{\alpha} : \mathbb{Z}G \to \mathbb{Z}[t^{\pm 1}]$, where M_n denotes the ring of matrices of order n. Now, let $G = \langle x_1, x_2, \ldots, x_m | r_1 = 1, r_2 = 1, \ldots, r_{m-1} = 1 \rangle$ be a Wirtinger presentation of $G(K)$. Let F be a free group freely generated by $\{x_1, x_2, \ldots, x_m\}$. Let $A = [\partial r_i / \partial x_j], 1 \leq i \leq m-1, 1 \leq j \leq m$, be an $(m-1) \times m$ matrix (over $\mathbb{Z}R$), where $\partial / \partial x_j$ denotes Fox's free derivatives. (See 8(1).) Now eliminate one column, say the last column to obtain $\hat{A}$. Let $\phi : F \to G$ be a natural homomorphism defined by $\phi(x_i) = x_i$ and $\phi : G \to M_n(R[t^{\pm 1}])$ be a homomorphism defined by $\phi(x_i) = \rho(x_i)t$. Extend ϕ to a ring homomorphism from $\mathbb{Z}G$ to $M_n(Z[t^{\pm 1}])$. Then $(\partial r_i / \partial x_j)^\phi \in M_n(R[t^{\pm 1}])$, and $\hat{A}^\phi = ||(\partial r_i / \partial x_j)^\phi||$ is an $n(m-1) \times n(m-1)$ matrix over $R[t^{\pm 1}]$. We define as in [Wad]:

$\Delta_{\rho,K}(t) = det \hat{A}^\phi / det(x_m^\phi - I)$, where I denotes the identity matrix of order n.

This is an invariant of a knot K up to $ut^{\pm k}$, where u is a unit of R.

Remark 13.1 [Wad] (1) If $\rho_o : G \to GL(n, \mathbb{Z})$ be a trivial representation, then $\Delta_{\rho_o,K}(t) = (\Delta_K(t)/(1-t))^n$. (2) $\Delta_{\rho,K}(t)$ is a rational function, but if there exists an element $w \in G'$ such that 1 is not an eigenvalue of $\rho(w)$, then $\Delta_{\rho,K}(t)$ is a Laurent polynomial with coefficients in the field of quotients R_0 of R. Therefore, $\Delta_{\rho,K}(t) \in R_0[t^{\pm 1}]$.

Example 13.1 Let K be a knot in Fig. 13.1. Then a Wirtinger presentation of $G(K)$ is given by $\langle x, y | xyxy^{-1}x^{-1}y^{-1} = 1 \rangle$. Then $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(y) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ define a linear representation. Then $\Delta_{\rho,K}(t) = (1-t)^2(1+t^2)/(1-t)^2 = 1 + t^2$.

(2) **Fibered knots**

As one of the general results, we mention the following theorem.

Theorem 13.1 [GKM] *If K is a fibred knot, then for any unimodular representation $\rho : G(K) \to SL(2n, F)$, $\Delta_{\rho,K}(t)$ is a rational function of monic polynomial, where F is a field.*

Remark 13.2 [Ki] If $F = \mathbb{C}$ or a finite field F_p, then $\Delta_{\rho,K}(t)$ is symmetric.

(3) **Twisted Alexander polynomial**

Now we consider the twisted Alexander polynomial $\Delta_{\rho,K}(t)$ of a 2-bridge knot $K = B(\alpha, \beta)$ associated to $\rho : G(K) \to SL(2, \mathbb{Z})$, where ρ is a canonical representation of $G(K)$. See Section 11. For convenience, we write a Wirtinger presentation of $G(K)$ as follows: $G(K) = \langle x, y | R = 1\rangle$, And assume that $\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(y) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ Then xy^{-1} belongs to G', but none of the eigenvalues of $\rho(xy^{-1})$ is 1, and hence by Remark 13.1 (2), $\Delta_{\rho,K}(t)$ is a Laurent polynomial over $\mathbb{Q}$. Further, for any prime p, ρ induces a representation $\rho_p : G(K) \to SL(2, \mathbb{Z}/p)$, and $\Delta_{\rho_p,K}(t) \equiv \Delta_{\rho,K}(t) \pmod p$, and therefore, by Remark 13.2, $\Delta_{\rho,K}(t)$ is an integer symmetric polynomial. Noting that $\Delta_{\rho,T(3,2)}(t) = 1 + t^2$, (see Example 13.1,) we see that $\Delta_{\rho,K}(t)$ is divisible by $1 + t^2$. Now, set $\lambda(t) = \Delta_{\rho,K}(t)/(1 + t^2)$. Then, by Theorem 13.1, we see that if $K = B(\alpha, \beta)$ is fibred, then $\lambda(t)$ is monic. However, for a 2-bridge knot, the converse also holds. Namely, we have

Theorem 13.2 [HiM2] *Suppose that the group of a 2-bridge knot $B(\alpha, \beta)$ has a canonical representation ρ on $SL(2, \mathbb{Z})$. Then, $B(\alpha, \beta)$ is fibred if and only if $\Delta_{\rho,K}(t)$ is monic.*

A table of the twisted Alexander polynomials of about two hundred 2-bridge knots with canonical representation ρ on $SL(2, \mathbb{Z})$ by Stoimenow reveals the following facts. Since these facts are not proven yet, we leave them as conjectures.

Conjecture 13.1 *Let K be a 2-bridge knot with a canonical representation ρ on $SL(2, \mathbb{Z})$. Then*

(1) $\lambda(1) = 1$. Therefore, $\lambda(t)$ is the Alexander polynomial of some knot $\tilde{K}$ in $\mathbb{R}^3$. What is $\tilde{K}$?

(2) $\lambda(-1) = q^2$ for some integer q.

14. Milnor's Invariants (I)

Let G be a finitely generated group, and let $\{G^{(q)}, q \geq 1\}$ be the lower central series of G, i.e. $G^{(1)} = G$ and $G^{(q)} = [G, G^{(q-1)}]$ for $q \geq 2$. In 1957, J. Milnor defined a sequence of numerical invariants for an n-component link L, $n \geq 2$, using the specific presentation of the nilpotent group $G(L)/G(L)^{(q)}$. These invariants are called *Milnor's $\bar{\mu}$-invariants.* These invariants are trivial for a knot, since $G^{(2)} = G^{(3)} = \ldots$, for the group of a knot. In this section, we give a definition of Milnor's $\bar{\mu}$-invariants and state a few properties of these invariants. In the next section, we will give a topological interpretation of these invariants.

(1) **Chen-Milnor presentation**

Consider a Wirtinger presentation of the group of an n-component link $L = K_1 \cup K_2 \cup \cdots \cup K_n$. See Section 2.

The set of generators is $\{x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{2,m_2}, \ldots, x_{n,1}, \ldots, x_{n,m_n}\}$, and a set of relations is $\{R_{1,1} = 1, \ldots, R_{1,m_1} = 1, R_{2,1} = 1, \ldots, R_{2,m_2} = 1, \ldots, R_{n,1} = 1, \ldots, R_{n,m_n} = 1\}$. Although one of relations is irrelevant, we keep it in the set.

Each relation is of the form, for $1 \leq i \leq n, 1 \leq j \leq m_i$, $R_{i,j} = x_{i,j} a_{i,j} x_{i,j+1}^{-1} a_{i,j}^{-1}$, where $a_{i,j} = x_{k,\ell}^{\pm}$ for some $k, \ell, 1 \leq k \leq n, 1 \leq \ell \leq m_k$. Now replace $R_{i,2}$, by $R'_{i,2} = R_{i,1}(a_{i,1} R_{i,2} a_{i,1}^{-1}) = x_{i,1}(a_{i,1}, a_{j,2}) x_{i,3}^{-1}(a_{i,2}^{-1} a_{i,1}^{-1})$, and then replace $R_{i,3}$ by $R'_{i,3} = R'_{i,2}(a_{i,1} a_{i,2} R_{i,3} a_{i,2}^{-1} a_{i,1}^{-1}) = x_{i,1}(a_{i,1} a_{i,2} a_{i,3})\ x_{i,4}^{-1}(a_{i,3}^{-1} a_{i,2}^{-1} a_{i,1}^{-1})$. Inductively, the set $\{R_{i,1}, R_{i,2}, \ldots, R_{i,m_i}\}$ is replaced by $\{R_{i,1}, R'_{i,2}, \ldots, R'_{i,m_i}\}$, where $R'_{i,j} = x_{i,j} W_{i,j} x_{i,j+1}^{-1} W_{i,j}^{-1}$, for $1 \leq i \leq n, 1 \leq j \leq m_i$, and $W_{i,j} = a_{i,1} a_{i,2} \cdots a_{i,j}$. In particular, $R'_{i,m_i} = x_{i,1} W_{i,m_i} x_{i,1}^{-1} W_{i,m_i}^{-1}$. Write $a_{i,j} = x_{k_{i,j} \ell_{i,j}}^{\varepsilon_{i,j}}$ and let $\lambda_i = \sum \varepsilon_{i,j}$, where the summation runs over all $a_{i,j}$ in W_{i,m_i} such that $k_{i,j} = i$. Then for

$i = 1, 2, \dots, n, \eta_i = W_{i,m_i} x_{i,1}^{-\lambda_i}$ represents a longitude of the ith component of K_i of L. Let F and F_n, respectively, be free groups that are freely generated, respectively, by $\{x_{i,j}, 1 \le i \le n, 1 \le j \le m_i\}$ and $\{x_1, x_2, \dots, x_n\}$. Now we define the homomorphisms $\theta_q : F \to F_n$ inductively.

(14.1) For $1 \le i \le n, 1 \le j \le m_i$ and $q \ge 1$, $x_{i,j}^{\theta_1} = x_i$, and $x_{i,1}^{\theta_{q+1}} = x_i, x_{i,j+1}^{\theta_{q+1}} = (W_{i,j}^{-1} x_{i,1} W_{i,j})^{\theta_q}$.

Then the following theorem is proved by Milnor [Mi] [C]:

Theorem 14.1 *For* $q \ge 1, G(L)/G(L)^{(q+1)}$ *has a presentation:* $\langle x_1, x_2, \dots, x_n | \eta_i^{\theta_q} x_i \eta_i^{-\theta_q} x_i^{-1} = 1, F_n^{(q+1)}, 1 \le i \le n\rangle$, *where* $F_n^{(q+1)}$ *denotes the* $(q+1)$*th member of the lower central series of* F_n.

(2) **Milnor's $\bar{\mu}$ - invariants**

Obviously, θ_q can be extended to the ring homomorphisms $\mathbb{Z}F$ to $\mathbb{Z}F_n$. The trivializer $o : \mathbb{Z}F_n \to \mathbb{Z}$ is a homomorphism defined by $\sum(a_i x_{i_1} \dots x_{i_p})^o = \sum a_i$. Now following Milnor [Mi], we define an integer $\mu(i_1 i_2 \dots i_p k)$ for a sequence of integers $\{i_1, i_2, \dots, i_p, k\}, p \ge 1, 1 \le i_1, i_2, \dots, i_p, k \le n$, as follows:

(14.2) $\mu(i_1 i_2 \dots i_p k) = (\partial^p \eta_k^{\theta_p} / \partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_p})^o$.

Let $\Delta(i_1 i_2 \dots i_r) = \gcd \mu(j_1 j_2 \dots j_s)$, where $j_1 j_2 \dots j_s, (2 \le s \le r)$ is to arrange over all sequences obtained by canceling at least one of the indices $i_1, i_2, \dots, i_r$ and permuting the remaining indices cyclically. Then Milnor proved:

Theorem 14.2 [Mi] $\bar{\mu}(i_1 i_2 \cdots i_p k) \equiv \mu(i_1 i_2 \dots i_p k) \pmod{\Delta(i_1 i_2 \cdots i_p k)}$ *is an invariant of* L.

This is called *Milnor's $\bar{\mu}$-invariant.*

Remark 14.1 (1) $\bar{\mu}(j_1 j_2 \cdots j_r)$ is in fact an isotopy invariant of L, where two n-component link L and L' are said to be *isotopic* if there is a continuous family of homeomorphisms $\{f_t, 0 \le t \le 1\}, f_t : L \to \mathbb{R}^3$ (or S^3) such that f_0 is an identity map and $f_1(L) = L'$.

(2) We define $\mu(i) = 0$ for any i.

(3) For $i \ne j, \mu(ij) = \mathrm{lk}(K_i, K_j)$, and $\mu(ii) = 0$. Due to these facts, Milnor's $\bar{\mu}$-invariants are sometimes called a *higher linking invariant.*

(4) $\bar{\mu}(\underbrace{ii \cdots i}_{k \text{ times}} j) \equiv \binom{\mu(ij)}{k} \mod \Delta(\underbrace{ii \dots i}_{k \text{ times}} j)$.

(5) If a sequence $\{i_1, i_2, \dots, i_r\}$ does not contain $j_1, j_2, \dots, j_k$, then $\bar{\mu}(i_1 i_2 \cdots i_r)$ is equal to Milnor's invariant for the sublink L' that excludes k components, $K_{j_1}, K_{j_2}, \dots, K_{j_k}$ from L.

(6) For a trivial n-component link, all Milnor's invariants vanish.

These invariants satisfy many relations. Some important relations are listed in the following theorem.

Theorem 14.3 [Mi] *(1) cyclic symmetry:* $\bar{\mu}(i_1 i_2 \cdots i_r) = \bar{\mu}(i_2 i_3 \cdots i_r i_1)$ *(2) shuffle equality: If* $i_1, \dots, i_r$ *and* $j_1, \dots, j_s$ *are given sequences with* $r, s \ge 1$, *then* $\sum \bar{\mu}(h_1 \cdots h_{r+s} k) \equiv 0 \mod \gcd \Delta(h_1 \cdots h_{r+s} k)$, *where the summation runs over all proper shuffles of* $i_1, \dots, i_r$ *and* $j_1, \dots, j_s$. *(A proper shuffle* $h_1, \dots, h_{r+s}$ *of* $i_1, \dots, i_r$ *and* $j_1, \dots, j_s$ *is one of the* $\binom{r+s}{s}$ *sequences obtained by intermeshing* $i_1, \dots, i_r$ *with* $j_1, \dots, j_s$.*)*

Like 2-bridge links or closed pure n-braids, if the group of an n-component link L is generated by exactly n Wirtinger generators, then $G(L)$ has a presentation: $G(L) = \langle x_1, x_2, \dots, x_n | \eta_i x_i \eta_i^{-1} x_i^{-1} = 1, 1 \le i \le n\rangle$. For this case, θ_q becomes an identity, and hence, $G(L)/G(L)^{(q+1)}$ has a presentation:

$\langle x_1, x_2, \ldots, x_n | \eta_i x_i \eta_i^{-1} x_i^{-1} = 1, F_n^{(q+1)}, 1 \leq i \leq n\rangle$. Therefore, the computation of $\bar{\mu}$-invariants is much simpler.

(3) **Examples**

Example 14.1 Let L be a link depicted in Fig. 14.1. For simplicity, we use x_i, y_j for $x_{1,i}, x_{2,j}$, and R_i, S_j for $R_{1,i}, R_{2,j}$. (In Fig. 14.1, we do not distinguish between an over-passing arc and a generator of $G(K)$.) Then $G(L) = \langle x_i, y_j, | R_i = 1, S_j = 1, i = 1,2,3, j = 1,2,\ldots,5\rangle$, where $R_1 = x_1 y_1 x_2^{-1} y_1^{-1}$, $R_2 = x_2 y_3 x_3^{-1} y_3^{-1}$, $R_3 = x_3 y_5 x_1^{-1} y_5^{-1}$, and $S_1 = y_1 y_5 y_2^{-1} y_5^{-1}$, $S_2 = y_2 x_1 y_3^{-1} x_1^{-1}$, $S_3 = y_3 y_5 y_4^{-1} y_5^{-1}$, $S_4 = y_4 x_1 y_5^{-1} x_1^{-1}$, $S_5 = y_5 x_1 y_1^{-1} x_1^{-1}$. The set of new relations $R_2', R_3', S_2', \ldots, S_5'$ are given below:

$R_2' = x_1(y_1 y_3) x_3^{-1}(y_3^{-1} y_1^{-1})$, $R_3' = x_1(y_1 y_3 y_5) x_1^{-1}(y_5^{-1} y_3^{-1} y_1^{-1})$, and hence, $\eta_1 = y_1 y_3 y_5$, and $S_2' = y_1(y_5 x_1) y_3^{-1}(x_1^{-1} y_5^{-1})$, $S_3' = y_1(y_5 x_1 y_5) y_4^{-1}(y_5^{-1} x_1^{-1} y_5^{-1})$, $S_4' = y_1(y_5 x_1 y_5 x_1)\ y_5^{-1}\ (x_1^{-1} y_5^{-1} x_1^{-1} y_5^{-1})$, $S_5' = y_1(y_5 x_1 y_5 x_1^2) y_1^{-1}(x_1^{-2} y_5^{-1} x_1^{-1} y_5^{-1})$, and hence, $\eta_2 = y_5 x_1 y_5 x_1^2 y_1^{-2}$. Now we evaluate some of $\bar{\mu}$-invariants. $\bar{\mu}(1) = \bar{\mu}(2) = 0$. To evaluate higher $\bar{\mu}$-invariants, first we compute θ_p. They are: $\theta_1(x_i) = x_1$ and $\theta_1(y_i) = y_1$, $\theta_2(x_1) = x_1$, $\theta_2(x_2) = y_1^{-1} x_1 y_1$, $\theta_2(x_3) = y_1^{-2} x_1 y_1^2$, and $\theta_2(y_1) = y_1, \theta_2(y_2) = y_1$, $\theta_2(y_3) = x_1^{-1} y_1 x_1$, $\theta_2(y_4) = (y_1^{-1} x_1^{-1}) y_1 (x_1 y_1)$, $\theta_2(y_5) = (x_1^{-1} y_1^{-1} x_1^{-1}) y_1 (x_1 y_1 x_1)$.

Using these formulas, we have $\eta_1^{\theta_1} = y_1^3$, $\eta_1^{\theta_2} = y_1 x_1^{-2} y_1 x_1 y_1 x_1$, and $\eta_2^{\theta_1} = y_1 x_1 y_1 x_1^2 y_1^{-1}$, $\eta_2^{\theta_2} = x_1^{-1} y_1^{-1} x_1^{-1} y_1 x_1 y_1 x_1 y_1^{-1} x_1^{-1} y_1 x_1 y_1 x_1^3 y_1^{-2}$. Now we are in position to evaluate $\bar{\mu}$-invariants. $\bar{\mu}(12) = (\partial \eta_2^{\theta_1} / \partial x_1)^o = 3$, $\bar{\mu}(21) = (\partial \eta_1^{\theta_1} / \partial y_1)^o = 3$, $\bar{\mu}(121) = (\partial^2 \eta_1^{\theta_2} / \partial x_1 \partial y_1)^o = 0$, since $\Delta(121) = 3$, $\bar{\mu}(112) = (\partial^2 \eta_2^{\theta_2} / \partial x_1^2)^o = 0$, since $\Delta(112) = 3$, $\bar{\mu}(122) = (\partial^2 \eta_2^{\theta_2} / \partial x_1 \partial y_1)^o = 0$, since $\Delta(122) = 3$. Similarly, we can show that $\bar{\mu}(1122) = 2$, since $\Delta(1122) = 3$. Further, by using Theorem 14.3, we have $\bar{\mu}(1212) = 2$. Therefore, $\Delta(j_1 j_2 j_3 j_4 j_5) = 1$, and $\bar{\mu}(i_1 i_2 \cdots i_k) = 0$ for $k \geq 5$.

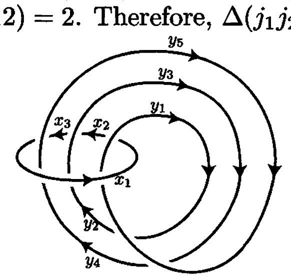


Figure 14.1

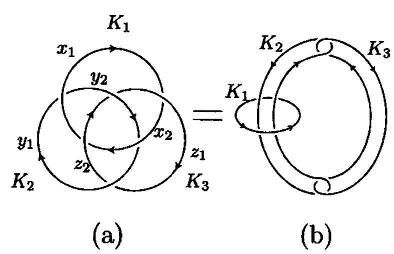


Figure 14.2

Example 14.2 Let L be a link depicted in Fig. 14.2 (a) or (b). This link is called the *Borromean rings*. (This link has a property that a removal of one arbitrary component results a trivial link.) As before, we use x_i, y_j, z_k for $x_{1,i}, x_{2,j}, x_{3,k}$, and R_i, S_j, U_k for relations. The group $G(L)$ has the following presentation:

$\langle x_1, x_2, y_1, y_2, z_1, z_2 | R_1 = 1, R_2 = 1, S_1 = 1, S_2 = 1, U_1 = 1, U_2 = 1\rangle$, where $R_1 = x_1 z_1^{-1} x_2^{-1} z_1, R_2 = x_2 z_2 x_1^{-1} z_2^{-1}, S_1 = y_1 x_1^{-1} y_2^{-1} x_1, S_2 = y_2 x_2 y_1^{-1} x_2^{-1}, U_1 = z_1 y_1^{-1} z_2^{-1} y_1, U_2 = z_2 y_2 z_1^{-1} y_2^{-1}$. And new relations are:
$R_2' = x_1(z_1^{-1} z_2) x_1^{-1}(z_2^{-1} z_1), S_2' = y_1(x_1^{-1} x_2) y_1^{-1}(x_2^{-1} x_1), U_2' = z_1(y_1^{-1} y_2) z_1^{-1}(y_2^{-1} y_1)$, and $\eta_1 = z_1^{-1} z_2, \eta_2 = x_1^{-1} x_2, \eta_3 = y_1^{-1} y_2$. Now, $\bar{\mu}(12) = \bar{\mu}(13) = \bar{\mu}(23) = 0$, and $\bar{\mu}(123) = 1, \bar{\mu}(132) = -1$. Therefore, $\Delta(j_1 j_2 j_3 j_4) = 1$, and $\bar{\mu}(i_1 i_2 \cdots i_k) = 0$ for $k \geq 4$.

(4) **Lower central quotients**

Let $G = G(L)$ be the group of an n-component link. If all $\bar{\mu}$-invariants vanish, then it is known that for any $q \geq 1, G/G^{(q)} \cong F/F_n^{(q)}$. However, if $\bar{\mu}(ij) = \pm 1$ for any i and $j, i \neq j$, then $\bar{\mu}(i_1 i_2 \dots i_k) = 0$ for $k \geq 3$. In this case, the following theorem is proved.

Theorem 14.4 *Let G be the group of an n-component link. Suppose that $\bar{\mu}(ij) = \pm 1$ for any i and $j, i \neq j$. Then for any $q \geq 2, G^{(q)}/G^{(q+1)} \cong F_{n-1}^{(q)}/F_{n-1}^{(q+1)}$, where F_{n-1} is a free group of rank $n-1$.*

This theorem was first proved by Kojima for the case $\bar{\mu}(ij) = 1$ for any i and $j, i \neq j$ [Ko]. Later, Maeda proved this theorem under weaker conditions [Ma], also see [MaT] [La]. In 2000, Morishita proved a number theoretical version of this theorem [Mo].

15. Milnor's invariants (II)

Milnor's $\bar{\mu}$-invariants are interpreted as covering linkage invariants in a certain covering space. Since $\bar{\mu}(i_1 i_2 \cdots i_r)$ is defined via a nilpotent group $G/G^{(q)}$, the covering will be most likely a nilpotent covering.

(1) **Nilpotent Representation**

To find an appropriate covering, first we define a free group action on $\mathbb{Z}^k$, the set of k-tuples $\{(a_1, a_2, \dots, a_k) | a_i \in \mathbb{Z}\}$. Let F be a free group of rank n freely generated by $x_1, x_2, \dots, x_n$. Given a sequence of integers $\xi = \{j_1, j_2, \dots, j_k\}$, $1 \leq j_1, j_2, \dots, j_k \leq n$, we define an action Φ of F on $\mathbb{Z}^k$: $\Phi_\xi(x_i)(a_1, a_2, \dots, a_k) = (b_1, b_2, \dots, b_k)$, where $b_1 = a_1 + \delta_{i,j_1}, b_2 = a_2 + \delta_{i,j_2} a_1, \cdots, b_k = a_k + \delta_{i,j_k} a_{k-1}$. For example, if $\xi = \{1, 2, 1, 1, 2\}$, then $\Phi_\xi(x_1)(a_1, a_2, \dots, a_5) = (a_1 + 1, a_2, a_3 + a_2, a_4 + a_3, a_5), \Phi_\xi(x_2)(a_1, a_2, \dots, a_5) = (a_1, a_2 + a_1, a_3, a_4, a_5 + a_4), \Phi_\xi(x_i) = \text{id.}$ for $i \geq 3$.

This action is written in the matrix form as follows:

$$\Phi_\xi(x_i) = \begin{bmatrix} 1 & \delta_{i,j_1} & & & \\ & 1 & \delta_{i,j_2} & & \text{\Large 0} \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & 1 & \delta_{i,j_k} \\ & & & & 1 \end{bmatrix} \in GL(k+1, \mathbb{Z}).$$

Let Ω be the orbit of $(0, 0, \dots, 0)$ under Φ_ξ and $S(\Omega)$, the group of permutations on Ω. Then we have a homomorphism $\Phi_\xi : F \to S(\Omega)$. Using this (transitive) homomorphism, we can define a permutation representation of $G(L)$. Let $\hat{\xi} = \{j_1, j_2, \dots, j_k, j_{k+1}, j_{k+2}\}$ be a sequence of the length $k+2$.

Let $m = \Delta(j_1 j_2 \cdots j_k j_{k+1} j_{k+2})$. Now we consider a nilpotent representation $\hat{\Phi}_\xi : G(L) \to S((\mathbb{Z}/m)^k)$, where $S((\mathbb{Z}/m)^k)$ denotes the group of permutations on k-tuples $\{(a_1, a_2, \dots, a_k) | 0 \leq a_i \leq m-1\}$.

(2) **Covering linkage invariants**

Let L be an n-component link in S^3, and M_Φ the covering space corresponding to $\hat{\Phi}_\xi$. Let $\tilde{L}$ be the lift of L in M_Φ. Assume $m \neq 0$. Let $\{\mathcal{Q}_0^{(p)}, \mathcal{Q}_1^{(p)}, \dots, \mathcal{Q}_\lambda^{(p)}\}$ be the set of all orbits in Ω under the actions $\hat{\Phi}_\xi(x_{p,1})$ and $\hat{\Phi}_\xi(\eta_p), p = j_1, j_2, \dots, j_{k+2}$. Then each orbit $\mathcal{Q}_i^{(p)}$ corresponds to one and only one component of $\tilde{L}$, denoted by $\tilde{K}_{p,i}$.

Now to each orbit $\mathcal{Q}_i^{(j_{k+1})}$, we define $q_i = \sum \beta_k(b_1, \dots, b_k)$, where the summation runs over all $(b_1, \dots, b_k)$ in $\mathcal{Q}_i^{(j_{k+1})}$ and $\beta_k(b_1, \dots, b_k) = b_k$. Then set $\alpha = \sum_{i=1}^{\lambda} q_i \tilde{K}_{j_{k+1},i} \in H_1(\tilde{L}; \mathbb{Z})$.

It is known that α bounds a 2-chain B in M_Φ and we have:

Theorem 15.1 [Mu6] *Given a sequence* $\hat{\xi} = \{j_1, j_2, \ldots, j_k, j_{k+1}, j_{k+2}\}$, *let* $\tilde{K}_{j_{k+2},0}$ *be the knot corresponding to the orbit* $Q_0^{(j_{k+2})}$ *that contains* $(0,0,\ldots,0)$. *Assume that* $m \neq 0$. *Then* $\bar{\mu}(j_1, j_2, \ldots, j_{k+2}) \equiv \text{int}\,(\tilde{K}_{j_{k+2},0}, B) \pmod{m}$, *where* $int(\, , \,)$ *stands for the intersection number in* M_Φ. *Therefore, if* $\tilde{K}_{j_{k+2},0}$ *is homologous to* 0 *in* M_Φ, *then* $\bar{\mu}(j_1, j_2, \ldots, j_{k+2}) \equiv \text{lk}(\tilde{K}_{j_{k+2},0}, \alpha) \pmod{m}$.

Remark 15.1 (1) If $m = 0$, then consider a finite nilpotent representation $\hat{\Phi}_\xi : G(L) \to S((\mathbb{Z}/q)^k)$ for a sufficiently large $q > 0$. Then 'mod q' will be interpreted as 'equality'.
(2) By Cyclic Symmetry (Theorem 14.3), we may assume without loss of generality that $j_{k+1} \neq j_{k+2}$.

Example 15.1 For $\xi = \{i, j, k\}$, we have from Theorem 15.1 that $\bar{\mu}(ijk) \equiv int(\tilde{K}_{k,0}, B) \pmod{\bar{\mu}(jk)}$, where $\partial B = \sum p\tilde{K}_{j,p} \in H_1(\tilde{L}, \mathbb{Z})$, and $\tilde{L}$ is a lift of L in the m-fold cyclic covering space of K_i.

(3) **Examples**

Example 14.1 (continued) First we see that $\bar{\mu}(12) = 3$. Since $\Delta(1jk) = 3$, M_Φ is the 3-fold cyclic covering space of K_1, and $M_\Phi \approx S^3$. Then the lift of L in M_Φ is depicted in Fig. 15.1.

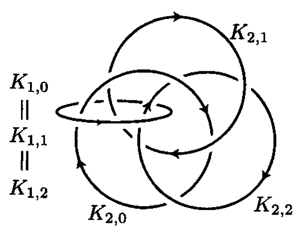


Figure 15.1

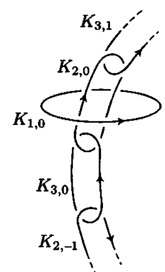


Figure 15.2

From this diagram, we can see easily that $\bar{\mu}(112) \equiv \text{lk}(K_{2,0}, K_{1,1} + 2K_{1,2}) = 3 \equiv 0 \pmod 3$, $,\bar{\mu}(121) \equiv \text{lk}(K_{1,0}, K_{2,1} + 2K_{2,2}) = 3 \equiv 0 \pmod 3, \bar{\mu}(122) \equiv \text{lk}(K_{2,0}, K_{2,1} + 2K_{2,2}) = 3 \equiv 0 \pmod 3$.

Example 14.2 (continued) Since $\bar{\mu}(12) = \bar{\mu}(13) = \bar{\mu}(23) = 0$, we see $\Delta(ijk) = 0$. To see $\bar{\mu}(123)$, consider the lift $\tilde{L}$ of L in the infinite cyclic covering space of K_1 (or the m-fold cyclic covering space with a sufficiently large m). $\tilde{L}$ is depicted in Fig. 15.2. Let $\alpha = \sum_i K_{2,i}$ and $\beta = \sum_i K_{3,i}$. Then we see $\bar{\mu}(123) = \text{lk}(K_{3,0}, \alpha) = \text{lk}(K_{3,0}, -K_{2,-1}) = 1, \bar{\mu}(132) = \text{lk}(K_{2,0}, \beta) = \text{lk}(K_{2,0}, K_{3,-1}) = -1$.

16. Questions and Problems

In this last section, we discuss some problems in knot theory related to the number theory.

(1) **Structure of the homology group**

Let M_Φ be the double covering space of an n-component link $L = K_1 \cup K_2 \cup \cdots \cup K_n$. Suppose $|H_1(M_\Phi; \mathbb{Z})| < \infty$. Then the 2-sylow subgroup Γ of $H_1(M_\Phi; \mathbb{Z})$ is of the form: $\Gamma = \mathbb{Z}/2^{p_1} + \mathbb{Z}/2^{p_2} + \cdots + \mathbb{Z}/2^{p_{n-1}}, p_i \geq 1$. Define $e_k(L) = |\{p_i | p_i \geq k\}|$. Then we have $e_1(L) = n - 1$ [Sa1]. Further, $e_2(L)$ is determined as follows. Let Λ be an $n \times n$ matrix over $\mathbb{Z}/2$:

$$\Lambda = \begin{bmatrix} \mu_{11} & \mu(12) & \cdots & \mu(1n) \\ \mu(12) & \mu_{22} & \cdots & \mu(2n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu(n1) & \mu(n2) & \cdots & \mu_{nn} \end{bmatrix} \pmod 2,$$

where $\mu(ij) = \mu(ji) = \mathrm{lk}(K_i, K_j), i \neq j$, and $\mu_{kk} = -\sum_{i=1, i\neq k}^{m} \mu(ki)$.

Then $e_2(L) = n - 1 - \operatorname{rank} \Lambda$.

Question 16.1 Can $e_k(L), k \geq 3$, be determined by Milnor's $\bar{\mu}$-invariants ?

Question 16.2 Given a sequence of positive integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m, (m \geq 1)$. Does there exist a link L in S^3 such that $e_1(L) = \lambda_1, e_2(L) = \lambda_2, \ldots, e_m(L) = \lambda_m$?

Remark 16.1 Question 16.1 has been solved completely by Hillmann, Matei and Morishita [HMM].

(2) **Galois group of the Alexander polynomials**

Let $\Delta_K(t)$ be the Alexander polynomial of a knot K.

Question 16.3 What is the Galois group of $\Delta_K(t)$?

Since $\Delta_K(t)$ is characterized by two conditions, see Theorem 7.2, this problem is to characterize the Galois group of the polynomials satisfying these conditions.

(3) **Hilbert class field**

(i) **Periodic knots**

We say that a knot K has *period p* if there exists an orientation preserving homeomorphism ϕ from S^3 to S^3 such that (a) $\phi(K) = K$, (b) $\phi^p = \mathrm{id}$, but $\phi^q \neq \mathrm{id}.$ for $0 < q < p$, (c) $\mathrm{Fix}(\phi)$ and K are disjoint, where $\mathrm{Fix}(\phi)$ denotes the set of points in S^3 that are fixed under ϕ. (We know that $\mathrm{Fix}(\phi)$ is a trivial knot.)

Example 16.1 The knot K depicted in Fig. 16.1 has period 3. In fact, ϕ is a rotation about the z-axis through $\frac{2}{3}\pi$, and obviously $\mathrm{Fix}(\phi)$ is the z-axis. This knot K also has period 2. More generally, a 2-bridge knot has period 2, and a torus knot of type (p, q) has periods p and q. A non-trivial knot can have only finitely many periods. The set of periods that a knot can have is certainly a knot invariant, but there is no algorithm to determine all periods of a knot.

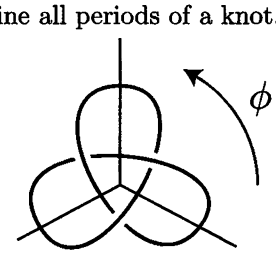


Figure 16.1

(ii) **Criteria on periodic knots**

The first result on the periodic knots was obtained by Trotter in 1960.

Theorem 16.1 [T1] *Suppose a knot K has period n. If (1)K is a fibred knot, and (2)$\Delta_K(t)$ has no repeated roots, then the splitting field F_Δ of $\Delta_K(t)$ over $\mathbb{Q}$ contains a primitive nth root of* 1.

Later, using the theory of the covering space, the following theorem was proved without any restriction.

Theorem 16.2 [Mu4] *Suppose a knot K has a period p, where p is a prime. Then*

$$\Delta_K(t) \equiv (1 + t + t^2 + \cdots + t^{\lambda-1})^{p-1} f(t)^p \pmod{p},$$

where $\lambda \geq 1$ and $f(t)$ is the Alexander polynomial of some knot.

The number λ and $f(t)$ are interpreted as follows. Suppose a knot K has period p, p a prime. Let $K_o = \mathrm{Fix}(\phi)$. Then S^3 is the p-fold cyclic covering space of a 3-manifold Σ branched along K_o, where $\Sigma \approx S^3$. Let $\psi : S^3 \to \Sigma$ be the projection. Then $\psi(K) = \hat{K}$ is a knot in Σ and it is shown that $f(t) = \Delta_{\hat{K}}(t)$, and further $\lambda = |\mathrm{lk}(K_o, \hat{K})| \neq 0$. (If $\lambda = 0$, then $\psi^{-1}(\hat{K})$ would be a link (not a knot). Often $\hat{K}$ is unknotted, so $f(t) = 1$.

(3) **Heilbronn's Conjecture**

By looking at above two theorems from the number theoretical points of view, Heilbronn suggested the following conjecture:

Conjecture *Let K be a knot. Suppose p is a prime. Then, if $\Delta_K(t) \equiv (1+t+t^2+\cdots+t^{\lambda-1})^{p-1} \pmod{p}$, $\lambda \geq 1$, then Hilbert class field $\hat{F}$ of F_Δ, the splitting field of $\Delta_K(t)$ over $\mathbb{Q}$, contains a primitive pth roof of* 1.

This conjecture is in general not true. The simplest example is the following

Example 16.2 Consider $\Delta_K(t) = 1 - 6t + 11t^2 - 6t^3 + t^4 (= (1 - 3t + t^2)^2) \equiv (1+t)^4 \pmod{5}$. Then $F_\Delta = \mathbb{Q}(\sqrt{5}) \in \mathbb{R}$, but $\mathbb{Q}(\sqrt{5})$ has class number 1 and hence $\hat{F} = F_\Delta$ does not contain a primitive 5th root of 1.

However, for some knot, the conjecture holds.

Example 16.3 Let $\Delta_K(t) = 4 - 7t + 4t^2 \equiv (1+t)^2 \pmod{3}$. Then $F_\Delta = \mathbb{Q}(\sqrt{-15})$ and $F_\Delta(\omega), \omega$ a primitive 3rd root of 1, is unramified over F_Δ, and hence $\hat{F}$ contains ω.

(3) **Revised Problem**

These two examples suggest that the conjecture should be revised, and Morishita now proposes the following question:

Question 16.4 (Morishita) Let p be a prime. Which Alexander polynomial $\Delta_K(t)$ of degree $p-1$ has the splitting field F_Δ over $\mathbb{Q}$ so that $F_\Delta(\xi), \xi$ a primitive pth root of 1, is unramified over F_Δ ?

Finally, we mention one of the recent result due to Morishita and Taguchi that gives a sufficient condition for $\Delta_K(t)$ to hold Heilbronn Conjecture.

Theorem 16.3 [MoT] *Let p be a prime $\neq 2$. Let $g(t)$ be an Eisenstein polynomial, i.e. $g(t) = t^{p-1} + a_1 t^{p-2} + \cdots + a_{p-1}$, where $p|a_i, 1 \leq i \leq p-1$, but $p^2 \nmid a_{p-1}$. Then, Hilbert class field of the splitting filed F of $g(t)$ contains a pth root of* 1.

Therefore, if $g(t+1) = \Delta_K(t)$ for some knot K, then $\Delta_K(t) \equiv (t+1)^{p-1} \pmod{p}$ and Hilbert class field of the splitting field of $\Delta_K(t)$ contains a primitive pth root of 1. For example, let $g(t) = t^4 + 5t^3 - 40t^2 + 70t - 35$. Then $g(t+1) = t^4 + 9t^3 - 19t^2 + 9t + 1 = \Delta_K(t)$ for some fibred knot K [Bu1], and $\Delta_K(t) \equiv (1+t)^4 \pmod{5}$. Since the splitting field F of $\Delta_K(t)$ does not contain a primitive 5th root of 1 (due to T. Komatsu), K does not have period 5 by Theorem 16.1, but by Theorem 16.3, Hilbert class field of F contains a primitive 5th root of 1.

Recently, using Theorem 16.1, T. Komatsu shows [Kom] that for every odd prime number, there are infinitely many knots satisfying Heilbronn's conjecture.

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