Mastermath Elliptic Curves, Homework 5
Due: 20th October 2015, 10:15

Students are expected to solve all problems below, except those marked as "optional". All problems (except those marked as optional) and their solutions are part of the course and could play a role in the exam. Only those marked as "hand in" (that is, 29, 31, and 33) are to be handed in and count towards your grade according to the rules on the web page.

The notation $\mathbb{F}_q$ denotes a field with q elements (for any prime power q).

Problem 28. Let k be a field. Let C be the smooth curve in $\mathbb{A}^2$ given by $y^2 = x$. Let $P = (\alpha, \beta)$ be a point in $C(k)$.

(a) Show that the function $y - \beta$ is a uniformizer of P

(b) Suppose that $\mathrm{char}(k) \neq 2$. Show that $x - \alpha$ is a uniformizer of P if and only if $P \neq (0, 0)$.

Problem 29. (hand in) Let k be a field and let C be the smooth projective curve given by the equation $Y^2 = XZ$ in $\mathbb{P}^2$. Let $P = (0 : 0 : 1)$ and $Q = (1 : 0 : 0)$.

(a) Show that the divisor of the function $f = Y/Z$ is equal to $P - Q$.

(b) Show that the divisor of the function $g = X/Z$ is equal to $2P - 2Q$.

(c) Exhibit a function whose divisor is $R - P$ where $R = (1 : 1 : 1)$.

Problem 30. Let k be a field of characteristic ≥ 5, let $A, B \in k$ be such that $4A^3 + 27B^2 \neq 0$, and let E be the elliptic curve over k given by the equation $y^2 = x^3 + Ax + B$. Let P be a point on E that is defined over k.

(a) Show that if P is of the form (α, β) with $\beta \neq 0$, then the function $x - \alpha$ is a uniformizer for P.

(b) Show that if P is of the form $(\alpha, 0)$, then the function y is a uniformizer for P.

(c) Show that if P is the point at infinity, then the function x/y is a uniformizer for P.

Problem 31. (hand in) Let C be a smooth projective curve defined over a field k and let D be a divisor defined over k. You may assume that k is perfect.

(a) Show that $L(D) = \{f \in k(C)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ is a k-vector space. Its dimension is denoted by $l(D)$.

(b) Suppose that D' is a divisor of C defined over k that is *equivalent* to D, i.e., the classes of D and D' in $\mathrm{Pic}(C)$ are equal. Show that $\deg(D) = \deg(D')$. Show that $l(D) = l(D')$.

(c) Suppose that $k = \mathbb{F}_q$ is a finite field with q elements. Show that the number of *effective* divisors of C that are defined over k and are equivalent to D is equal to
$$\frac{q^{l(D)} - 1}{q - 1}.$$

Problem 32. Let k be a field. For convenience we assume that it is algebraically closed. Let C be a smooth projective curve defined over a field k and let $D = \sum_Q n_Q Q$ be a divisor of C.

(a) Show that $l(D) = 0$ if $\deg(D) < 0$.

(b) Let P be a point on C and let $n = \mathrm{ord}_P(D) := n_P$. Let $\pi \in k(C)$ be a uniformizer for P. In other words, we have $\mathrm{ord}_P(\pi) = 1$. Show that the map
$$L(D) \longrightarrow k$$
given by $f \mapsto (f/\pi^n)(P)$ is k-linear and show that its kernel is $L(D - P)$.

(c) Show that for all divisors D and all points P we have
$$\big(l(D) - l(D - P)\big) \in \{0, 1\}.$$
On other words, adding a point to a divisor will add at most one to the dimension of its linear system.

(d) Deduce from (a) and (c) that for every divisor D we have
$$l(D) \leq \max\{0, \deg D + 1\}.$$

Problem 33. (hand in) Let C be a smooth projective curve over a finite field $\mathbb{F}_q$.

(a) Let $d \geq 1$. Show that C has only finitely many points that are defined over $\mathbb{F}_{q^d}$.

(b) (optional: only for those who have taken a course on Galois theory) Conclude from (a) that for every integer d there are only finitely many effective divisors of degree d that are defined over $\mathbb{F}_q$.

You may use (b) even if you did not solve it.

(c) Show that every sufficiently large d the set of divisor classes of divisors of degree d that are defined over $\mathbb{F}_q$, is finite. Hint: use the Riemann-Roch theorem.

(d) Show that the group $\mathrm{Pic}^0(C)$ is finite.

Problem 34. (optional) Do problems 29 and/or 30 of `http://pub.math.leidenuniv.nl/~javanpeykara2/ec2013/EC2013HW3.pdf`, which is also the basis of the other problems on this homework set.

Problem 35. (optional) Let C be a smooth projective curve defined over a field k. For convenience we assume that it is algebraically closed. The support of a divisor $D = \sum_P n_P P$ is the finite set of points for which $n_P \neq 0$. For any divisor $D = \sum_P n_P P$ and any function $f \in k(C)^*$ for which the supports of D and $\mathrm{div}(f)$ are disjoint, we put $f(D) = \prod_P f(P)^{n_P}$. *Weil reciprocity* is the statement that

$$f(\mathrm{div}(g)) = g(\mathrm{div}(f)).$$

(a) Show that the divisor of the function $f = \prod_{i=1}^{m}(x - \alpha_i) \in k(\mathbb{P}^1)$ is $\sum_{i=1}^{m}(\alpha_i : 1) - m\ (1 : 0)$.

(b) Prove Weil reciprocity for for $C = \mathbb{P}^1$.

Problem 36. (optional: extra practise) Let k be a field with $\mathrm{char}(k) \neq 7$ and let K be the Klein curve: $X^3Y + Y^3Z + Z^3X = 0$ in $\mathbb{P}^2$.

(a) Show that K is smooth.

(b) Compute the divisors of the functions X/Y and X/Z.